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Finite Trotter Approximation to the Averaged Mean Square Distance in the Anderson Model

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Abstract We prove that a finite Trotter approximation to the averaged mean square distance traveled by a particle in a disordered system on a lattice $\mathbb{Z}^d$ exhibits at most a diffusive behavior in dimensions $d \geq 3$ as long as the Fourier transform of the single-site probability, $\hat{\mu}$, is in $L^2(\mathbb{R})$.

Keywords Anderson model · Random Schrödinger operators · Trotter product formula

1 Introduction

Localization has been a subject of high interest to mathematicians and physicists over the past several decades. One of the most often studied quantities is the averaged mean square distance traveled by a particle in initial state ψ up to time t. In a d-dimensional lattice $\mathbb{Z}^d$, this quantity is defined as

$$r_\psi(t) := \left\{ \mathbb{E} \sum_{\mathbf{x}\in\mathbb{Z}^d} |\mathbf{x}|^2 |[e^{-it\mathbf{H}}\psi](\mathbf{x})|^2 \right\}^{1/2}, \tag{1.1}$$

where $\mathbb{E}(\cdot)$ is the expectation taken with respect to the probability configuration. The precise definitions of all constituents will be given in Sect. 2. For large t, if $r_\psi(t) \sim ct$, we say that the motion is ballistic, like that of a free particle; if $r_\psi(t) \sim c\sqrt{t}$, we say that the motion is diffusive, and if $\sup_{t>0} r_\psi(t)$ is bounded, then we say that the wave packet is (*physically*) localized. The behavior of $r_\psi(t)$ for large t is important in solid-state and condensed-matter physics because it is connected to the conductivity of the system via Kubo's formula [22, 29].

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The study of localization started in 1958 when P.W. Anderson [4] first argued that sufficiently large disorder essentially permits no diffusion to take place and halts the propagation of a wave function through a random medium. In the context of electronic properties of the system, the presence of sufficiently large disorder converts a good conductor to an insulator. Since then, the study of localization or delocalization has been translated to many mathematical formulations; see [6, 7, 17, 19, 23] and references therein. There are multiple mathematical interpretations of localization, and thus, more than one approach to its study. One commonly taken approach is known as a *spectral approach*, and it concerns the spectral properties of the operator **H**. Specifically, in the spectral approach, localization, often termed Anderson localization, occurs when **H** exhibits a dense pure point spectrum with exponentially decaying eigenfunctions. Note that there are various definitions of localization related to the spectral properties of **H**. See [23], for example, for precise definitions of these variations. In addition, the spectral approach has been extended to study localization in other domains such as Bethe lattice, trees, strips or half-planes, see [3, 7, 14, 24–26, 33] and references therein for more details.

For historical notes, the pioneering work of mathematical localization in the spectral approach was done by the Russian school [7, 21, 30] who studied the spectral properties of random Schrödinger operators in one dimension in 1970's. However, the groundbreaking result in multidimensional localization was obtained in 1983, when J. Fröhlich and T. Spencer proved exponential decay of the Green's functions for large disorder and at fixed energy [7, 15, 16]. The exponential decay of the Green's functions, which implies $\lim_{t\to\infty} \frac{r_\psi(t)}{t} = 0$ for any wave function with energy localized around the fixed energy at time $t = 0$ and the absence of an absolutely continuous spectrum (see notes at the end of Chap. IX in [7] and reference therein), has essentially become a much sought property of **H**. Subsequently, most work done in this approach has been about or related to the spectral properties of **H**.

In one dimension, there are presently many proofs [7, 15, 16, 34, 38] that Anderson localization occurs for all disorder and at every energy. However, the techniques of these proofs do not carry over to higher dimensions for arbitrary disorder. In dimension $d = 2$, it is believed that **H** exhibits the same behavior as that in $d = 1$, but a proof is missing. In particular, there has been no proof as to whether the eigenfunctions should decay exponentially or polynomially for small disorder [7, 11]. For sufficiently large disorder or at extreme energies near the edges of the spectrum $\sigma(\mathbf{H}_0)$, Anderson localization has been proved in arbitrary dimensions [2, 7, 15, 16, 18, 22, 27, 29, 36].

Another approach to study localization (or delocalization) is known as a *dynamical approach*, in which one is interested in a physics-related quantity like $r_\psi(t)$. In this case, we say that dynamical localization occurs if $r_\psi(t)$ is bounded; otherwise, we say that dynamical delocalization occurs. See [6, 8, 12, 13, 17] for variations of definitions of dynamical localization. In this approach, very few results are known; see [1, 6, 17, 35] for some results on quantities similar to $r_\psi(t)$. In the large-disorder or extreme-energy regime, it has been proved in [15, 16, 28] that the mean square distance is bounded uniformly in t with probability one. It is worth noting that the existence of dynamical delocalization and mobility edge has been proved in a two dimensional random Landau model [19]. Furthermore, the spectral approach and the dynamical approach are not necessarily equivalent; in particular, spectral localization does not always imply dynamical localization [9].

Even though $r_\psi(t)$ has direct connections to actual physical quantities such as conductivity of the system, it is relatively less-studied in a mathematical setting. The behavior of $r_\psi(t)$ at small disorder is still an open problem, and it is expected to have a diffusive behavior for large t in dimensions $d > 2$.

This article is organized as follows. In the upcoming Sect. 2, we state our problem, its framework and the main result. In Sect. 3, we utilize the Fourier transform to reformulate the problem at hand in terms of oscillatory integrals which allows us to do our estimates in the momentum space. Section 4 is devoted to study oscillatory integrals and the properties of the phases after taking the expectation $\mathbb{E}_\omega(\cdot)$, with some generic estimates on oscillatory integrals with two phases included in Appendix B. Finally, and most importantly, the proof of the main result will be given in Sect. 5.

2 Statement of the Problem and the Main Result

A beginning step to show that $r_\psi(t)$ displays a diffusive behavior for large t and for small disorder in dimensions $d > 2$ is to investigate whether $r_\psi(t) \le c\sqrt{t}$ for any disorder. Our approach is employing the Trotter product formula to estimate the growth of $r_\psi(t)$, without invoking the spectral structure of $\mathbf{H}$.

2.1 Anderson Model and Statement of the Problem

Let $\Omega = \times_{\mathbf{x}\in\mathbb{Z}^d}\mathbb{R}$ denote a probability space equipped with the probability measure $d\mathbb{P}(\omega) = \prod_{\mathbf{x}\in\mathbb{Z}^d}\mu(dv_\mathbf{x})$. For each $\omega \in \Omega$, let $\mathbf{V}_\omega$ be a random-potential multiplication operator on the lattice $\mathbb{Z}^d$ defined as

$$(\mathbf{V}_\omega\varphi)(\mathbf{x}) = v_\omega(\mathbf{x})\varphi(\mathbf{x}), \tag{2.1}$$

where $\{v_\omega(\mathbf{x})\}_{\mathbf{x}\in\mathbb{Z}^d}$ is a collection of independent identically distributed (i.i.d.) random variables with the single-site probability distribution given by $\mu(dv)$. Let $\mathbf{H}_0$ denote a discrete Laplace operator defined on $\ell^2(\mathbb{Z}^d)$ as

$$(\mathbf{H}_0\varphi)(\mathbf{x}) := -\sum_{|j|=1}\varphi(\mathbf{x}+j). \tag{2.2}$$

Then the Anderson model is a random self-adjoint Schrödinger operator $\mathbf{H}_\omega := \mathbf{H}_0 + \mathbf{V}_\omega$ on a Hilbert space $\mathcal{H} = \ell^2(\mathbb{Z}^d)$.

Remark 1 Often the Anderson model is defined as $\mathbf{H}_\omega := \mathbf{H}_0 + \lambda\mathbf{V}_\omega$, where $\lambda \in \mathbb{R}$ is a coupling constant representing a disorder strength. When proving the upper bound on $r_\psi(t)$ for arbitrary disorder is the main objective, it is usually set $\lambda = 1$ and assumed that the single-site distribution has a bounded density $\mu \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$, with $\|\mu\|_{L^1} = 1$. In this case, a disorder strength is defined in terms of $\|\mu\|_\infty^{-1}$.

Let $\mathbf{X}$ be a multiplication operator defined as

$$(\mathbf{X}\varphi)(\mathbf{x}) = \mathbf{x}\varphi(\mathbf{x}) \tag{2.3}$$

for $\varphi \in \mathcal{D}(\mathbf{X}) \subset \ell^2(\mathbb{Z}^d)$.

Definition 1 Define

$$\mathbf{p} := i[\mathbf{H}_0, \mathbf{X}], \tag{2.4}$$

$$\mathbf{X}_\omega(t) := e^{it\mathbf{H}_\omega}\mathbf{X}e^{-it\mathbf{H}_\omega}, \tag{2.5}$$

$$\mathbf{p}_\omega(t) := e^{it\mathbf{H}_\omega}\mathbf{p}e^{-it\mathbf{H}_\omega}. \tag{2.6}$$

Explicitly, the operator $\mathbf{p}$ is given by

$$(\mathbf{p}\varphi)(\mathbf{x}) = -i \sum_{|j|=1} j\varphi(\mathbf{x}+j). \tag{2.7}$$

By definition, $r_\psi(t) = \{\mathbb{E}_\omega \|\mathbf{X}_\omega(t)\psi\|^2\}^{1/2}$, so our question is equivalent to whether

$$\frac{1}{t}\mathbb{E}_\omega \|\mathbf{X}_\omega(t)\psi\|^2 \leq \text{ constant}, \tag{2.8}$$

for $\psi \in \mathcal{D}(\mathbf{X})$ and large t, where $\mathbb{E}_\omega(\cdot)$ is the expected value taken with respect to $d\mathbb{P}(\omega)$.

Remark 2 Observations

(i) In this article, the operators $\mathbf{H}_0$, $\mathbf{p}$ and $\mathbf{X}$ do not depend on a realization $\omega \in \Omega$, while $\mathbf{V}_\omega, \mathbf{H}_\omega, \mathbf{X}_\omega(t)$ and $\mathbf{p}_\omega(t)$ do. Often, we will suppress the ω dependence on these operators.

(ii) $\mathbf{H}_0$ and $\mathbf{p}$ are bounded operators on $\ell^2(\mathbb{Z}^d)$ with $\|\mathbf{H}_0\| \leq 2d$ and $\|\mathbf{p}\| \leq \sqrt{2d}$, while $\mathbf{X}$ is not. All of the operators mentioned above are self-adjoint on $\ell^2(\mathbb{Z}^d)$. In particular, $\mathbf{X}_\omega(t)$ and $\mathbf{p}_\omega(t)$ are also self-adjoint for each $t \in \mathbb{R}$; and they are respectively called the Heisenberg position and momentum operators.

(iii) If $\mathbf{V}$ is constant, then $\mathbf{H}$ and $\mathbf{p}$ commute; hence, the inequality (2.8) is trivially false. It is also false when $\mathbf{V}$ is periodic [7].

(iv) Even though the operator $\mathbf{X}(t)$ is unbounded, its derivative $\mathbf{p}(t)$ is bounded by $\sqrt{2d}$ uniformly in t. Thus, we can write $\mathbf{X}(t)$ in an integral representation as

$$\mathbf{X}(t) = \mathbf{X} + \int_0^t \mathbf{p}(s)\, ds, \tag{2.9}$$

where the equality holds on the domain $\mathcal{D}(\mathbf{X})$ of $\mathbf{X}$.

2.2 Trotter Product Theorem

A key ingredient in our analysis is the Trotter product theorem which states:

Theorem 2.1 *Suppose A and B are self-adjoint operators on a Hilbert space $\mathcal{H}$, and suppose that $A+B$ is self-adjoint (or essentially self-adjoint) on $\mathcal{D} = \mathcal{D}(A) \cap \mathcal{D}(B)$. Then,*

$$e^{it(A+B)} = \lim_{N\to\infty} \left[e^{i\frac{t}{N}A} e^{i\frac{t}{N}B}\right]^N, \tag{2.10}$$

where the limit is taken in the strong-topology.

Proof The proof of this well-known theorem can be found in, for example, [32]. □

To prove (2.8) via a Trotter product approximation, one naturally asks whether the corresponding statement holds when $\mathbf{U}(t) := e^{it\mathbf{H}}$ is replaced by $\mathbf{U}_N(t) := [e^{i\frac{t}{N}\mathbf{V}} e^{i\frac{t}{N}\mathbf{H}_0}]^N$. This is the main purpose and content of this article. Here, $\mathbf{U}^*(t) := e^{-it\mathbf{H}}$ is the adjoint of $e^{it\mathbf{H}}$, thus will be replaced by $\mathbf{U}_N^*(t) := [e^{-i\frac{t}{N}\mathbf{H}_0} e^{-i\frac{t}{N}\mathbf{V}}]^N$.

Definition 2 (Trotter Product Approximations) Define

$$\mathbf{p}_N(t) := \mathbf{U}_N(t)\mathbf{p}\mathbf{U}_N^*(t), \tag{2.11}$$

$$\mathbf{X}_N(t) := \mathbf{X} + \int_0^t \mathbf{p}_N(s)\,ds. \tag{2.12}$$

2.3 Main Result

Throughout this article, $\mathfrak{F}$ and $\hat{}$ denote the Fourier transform, while both $\mathfrak{F}^{-1}$ and $\check{}$ denote the inverse Fourier transform. The notation $Q = O_N(t)$ means that $|Q| \le C(N)t$, where the N dependence of the coefficient is not determined. The main result proved in this report can be stated as follows.

Proposition 1 (Main Result) *Let $d \ge 3$. Suppose the single-site probability distribution μ is such that $\check{\mu} \in L^2(\mathbb{R})$. For any fixed but arbitrary $N < \infty$, let $\mathbf{X}_N(t)$ denote the approximated position operator defined in (2.12). Then, for large t,*

$$\mathbb{E}_\omega \|\mathbf{X}_N(t)\delta_\mathbf{0}\|^2 = O_N(t).$$

Remark 3 It should be noted that our technique also yields some bounds in dimensions $d = 1, 2$ which are worse than $O_N(t)$. See Corollary 7 in Appendix B.

Remark 4 The quantities $\|\mathbf{X}(t)\psi\|^2$ and $\mathbb{E}_\omega\|\mathbf{X}(t)\psi\|^2$ respectively denote the mean square distance and the averaged mean square distance of a particle in an initial state ψ up to time t. In a position space, one can intuitively expect that, for large N (e.g., $N > 3$), the particle is experiencing many collisions. Between two collisions, the particle can move freely, but it changes direction with each collision. This should result in many reflections, hence cancellations from interference, giving rise to a diffusive net effect. The Trotter product approximation should capture this random-walk-like nature. Since $\mathbf{X}$ is unbounded, but $\mathbf{p}$ is bounded, we have chosen to apply the Trotter product formula to $\mathbf{p}(t)$ because the convergence is guaranteed. In addition, using the Newton's equation before applying the Trotter product formula to $\mathbf{p}(t)$ should give a better approximation. This follows because the $\mathbf{p}(t)$ approximation takes into account interactions occurring at all times since there are integrals over the time, rather than just those at N times with large intervals of free motion in between.

2.3.1 A Remark on Absolutely Continuous Spectrum

In the closing of this section, we remark that while our problem concerns the "fastest" component of ψ, it is also interesting to consider the "slowest" component of ψ. In the following lemma, we prove that when, for some $p > 0$, $|\mathbf{x}|^p$ in the "slowest" component of ψ grows faster than t^α, with $\alpha > 1/2$, then the spectral measure associated with ψ is absolutely continuous.

Lemma 1 *Let $0 < p < \infty$. Suppose that $\psi \in \mathcal{D}(|\mathbf{X}|^p)$ and that*

$$\left\| \frac{1}{\langle \mathbf{X} \rangle^p} e^{-it\mathbf{H}} \psi \right\| \le c|t|^{-\alpha},$$

for some $c > 0$ and $\alpha > 1/2$. Then, the spectral measure μ_ψ associated with ψ is absolutely continuous.

Proof By the spectral theorem, $(\psi, e^{-it\mathbf{H}}\psi) = \int e^{-itx}\, d\mu_\psi(x) := \hat{\mu}_\psi(t)$. Therefore, by the Schwarz inequality, the assumptions imply that

$$|\hat{\mu}_\psi(t)| \leq \|\langle \mathbf{X}\rangle^p \psi\| \cdot c|t|^{-\alpha} \leq \tilde{c}|t|^{-\alpha},$$

which implies that $\hat{\mu}_\psi \in L^2(\mathbb{R})$. This means $d\mu_\psi(x) = f(x)dx$, for some $f \in L^2(dx)$. Therefore, μ_ψ is absolutely continuous with respect to Lebesgue measure. □

3 Equivalent Statements

3.1 Fourier Transform and Spectral Theory

The Fourier transform will be employed on a few occasions. With the spectral theory, we will utilize the Fourier transform to represent $e^{\pm it\mathbf{H}_0}$ on $\ell^2(\mathbb{Z}^d)$. On another occasion, the Fourier transform comes up when we take the expectation $\mathbb{E}_\omega(\cdot)$.

Definition 3 Let $\mu(dv)$ denote the single-site probability distribution. Then,

$$\mathbb{E}_\omega(e^{itv_\omega}) := \int e^{itv}\, d\mu(v) := \check{\mu}(t). \tag{3.1}$$

Definition 4 For $\varphi \in \ell^2(\mathbb{Z}^d)$, we define the Fourier transform and its inverse by

$$\hat{\varphi}(\kappa) := \sum_{\mathbf{x}\in\mathbb{Z}^d} e^{-i\kappa\cdot\mathbf{x}}\varphi(\mathbf{x}), \tag{3.2}$$

$$\varphi(\mathbf{x}) := \int_{\mathbb{T}^d} e^{i\kappa\cdot\mathbf{x}}\hat{\varphi}(\kappa)\, d\kappa, \tag{3.3}$$

where $d\kappa$ is understood as a normalized Lebesgue measure on a d-dimensional torus $\mathbb{T}^d := [-\pi, \pi]^d$.

It turns out that it is more convenient to work in the momentum space rather than in the position space. The following two lemmas give representations of the considered operators in the momentum space.

Lemma 2 *In the Fourier representation, for* $\varphi \in \ell^2(\mathbb{Z}^d)$,

$$\mathfrak{F}(\mathbf{H}_0\varphi)(\kappa) = \xi(\kappa)\hat{\varphi}(\kappa), \tag{3.4}$$

$$\mathfrak{F}(\mathbf{p}\varphi)(\kappa) = \mathfrak{p}(\kappa)\hat{\varphi}(\kappa), \tag{3.5}$$

with

$$\xi(\kappa) = -2\sum_{j=1}^{d}\cos(\kappa_j), \tag{3.6}$$

$$\mathfrak{p}(\kappa) = 2\sum_{j=1}^{d}\sin(\kappa_j)e_j, \tag{3.7}$$

where κ_j denotes the j-th component of $\kappa \in \mathbb{T}^d$, and e_j denotes a standard unit vector in the j-th direction in $\mathbb{Z}^d$.

Proof The proof follows from the definitions. □

Remark 5 Adding the diagonal term $2dI$ to the free Hamiltonian $\mathbf{H}_0$ will not affect $\mathfrak{p}(\kappa)$; however, the free energy function $\xi(\kappa)$ is translated to $\tilde{\xi}(\kappa) := 2d + \xi(\kappa)$.

Remark 6 $\xi(\kappa)$ can be decomposed into a sum of d identical factors; that is, $\xi(\kappa) = \sum_{l=1}^{d} \xi_l(\kappa_l)$, where $\xi_l(x) = -2\cos(x)$ is the free energy corresponding to the one-dimensional problem.

Define $G_{\pm t}(\kappa) := e^{\pm it\xi(\kappa)}$. Then, by the spectral theorem,

$$\mathfrak{F}(e^{\pm it\mathbf{H}_0}\psi)(\kappa) = G_{\pm t}(\kappa)(\mathfrak{F}\psi)(\kappa) = G_{\pm t}(\kappa)\hat{\psi}(\kappa). \tag{3.8}$$

Consequently, $e^{\pm it\mathbf{H}_0}\psi$ and $\mathbf{p}\psi$ can be equivalently expressed as

$$(e^{\pm it\mathbf{H}_0}\psi)(\mathbf{x}) = \int_{\mathbb{T}^d} e^{i\kappa\cdot\mathbf{x}} e^{\pm it\xi(\kappa)}\hat{\psi}(\kappa)\,d\kappa; \tag{3.9}$$

$$(\mathbf{p}\psi)(\mathbf{x}) = \int_{\mathbb{T}^d} e^{i\kappa\cdot\mathbf{x}} \mathfrak{p}(\kappa)\hat{\psi}(\kappa)\,d\kappa. \tag{3.10}$$

3.2 Trotter Product Approximation

The finite Trotter approximations to $\mathbf{U}(t)$ and $\mathbf{U}^*(t)$ are given, respectively, by

$$\mathbf{U}_N(t) := \left[e^{i\frac{t}{N}\mathbf{V}} e^{i\frac{t}{N}\mathbf{H}_0}\right]^N, \tag{3.11}$$

$$\mathbf{U}_N^*(t) := \left[e^{-i\frac{t}{N}\mathbf{H}_0} e^{-i\frac{t}{N}\mathbf{V}}\right]^N. \tag{3.12}$$

Applying (2.12) and the Fubini theorem, it follows that

$$\begin{aligned} r_{N,\psi}^2(t) &:= \mathbb{E}_\omega \|\mathbf{X}_N(t)\psi\|^2 \\ &= \|\mathbf{X}\psi\|^2 + \int_0^t \mathbb{E}_\omega\Big[(\mathbf{X}\psi, \mathbf{p}_N(s)\psi)\Big]\,ds + \int_0^t \mathbb{E}_\omega\Big[(\mathbf{p}_N(u)\psi, \mathbf{X}\psi)\Big]\,du \\ &\quad + \int_0^t\int_0^t \mathbb{E}_\omega\Big[(\psi, \mathbf{p}_N(s)\cdot\mathbf{p}_N(u)\psi)\Big]\,dsdu, \end{aligned} \tag{3.13}$$

for each $\psi \in \mathcal{D}(\mathbf{X})$.

Remark 7 Since $\mathbf{p}$ is bounded,

$$\begin{aligned} &\left|\int_0^t\int_0^t \mathbb{E}_\omega\Big[(\psi, \mathbf{p}_N(s)\cdot\mathbf{p}_N(u)\psi)\Big]\,dsdu\right| \\ &\quad\le c^2\|\mathbf{p}\|^2\|\psi\|^2 + 2ct\|\mathbf{p}\|^2\|\psi\|^2 + \left|\int_c^t\int_c^t \mathbb{E}_\omega\Big[(\psi, \mathbf{p}_N(s)\cdot\mathbf{p}_N(u)\psi)\Big]\,dsdu\right|, \end{aligned} \tag{3.14}$$

for any constant $c > 0$ and $\psi \in \ell^2(\mathbb{Z}^d)$.

Remark 8 Since $\mathbf{p}(t) := s.\lim_{N\to\infty} \mathbf{p}_N(t)$,

$$\mathbb{E}_\omega\Big((\psi, \mathbf{p}(s)\cdot\mathbf{p}(u)\psi)\Big) = \mathbb{E}_\omega\Big(\lim_{N\to\infty}(\psi, \mathbf{p}_N(s)\cdot\mathbf{p}_N(u)\psi)\Big)$$

$$= \lim_{N\to\infty}\mathbb{E}_\omega\Big((\psi, \mathbf{p}_N(s)\cdot\mathbf{p}_N(u)\psi)\Big),$$

by Lebesgue's dominated convergence theorem. As a result, for $\psi \in \mathcal{D}(\mathbf{X})$,

$$\mathbb{E}_\omega\|\mathbf{X}(t)\psi\|^2 = \lim_{N\to\infty}\mathbb{E}_\omega\|\mathbf{X}_N(t)\psi\|^2. \tag{3.15}$$

In accordance with (3.11) and (3.12), for each $t \in \mathbb{R}$, let $\mathbf{V}(t) := e^{it\mathbf{V}}$ and $\mathbf{W}(t) := e^{it\mathbf{H}_0}$ denote two unitary groups, where $\mathbf{V}(t)$ is also a multiplication operator given by

$$(\mathbf{V}(t)\psi)(\mathbf{x}) = e^{itv_\omega(\mathbf{x})}\psi(\mathbf{x}), \tag{3.16}$$

for each $\mathbf{x} \in \mathbb{Z}^d$, $\omega \in \Omega$ and $\psi \in \ell^2(\mathbb{Z}^d)$. For each $\psi \in \ell^2(\mathbb{Z}^d)$, the Fourier transforms of $\mathbf{W}(t)\psi$ and $\mathbf{p}\psi$ are given, respectively, in (3.6) and (3.7); however, the Fourier transform of $e^{itv_\omega(\mathbf{x})}$ does not exist. For $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{Z}^d$, let $\|\mathbf{x}\| = \sum_{j=1}^d |x_j|$ denote a $\mathbb{Z}^d$-norm of $\mathbf{x}$. For $\epsilon > 0$ and $t \in \mathbb{R}$, let $\mathbf{V}_\epsilon(t)$ denote a multiplication operator given by

$$(\mathbf{V}_\epsilon(t)\psi)(\mathbf{x}) := V_t(\mathbf{x})\psi(\mathbf{x})$$

where

$$V_t(\mathbf{x}) := e^{itv_\omega(\mathbf{x})}e^{-\epsilon\|\mathbf{x}\|} \tag{3.17}$$

is a function in $\ell^2(\mathbb{Z}^d)\cap\ell^1(\mathbb{Z}^d)$ whose Fourier transform exists. Then, the Fourier transform of $\mathbf{V}_\epsilon(t)\psi$ is given by

$$\mathfrak{F}(\mathbf{V}_\epsilon(t)\psi)(\kappa) = \mathfrak{F}(V_t\psi)(\kappa) = \int_{\mathbb{T}^d}\hat{V}_t(\kappa-\eta)\hat{\psi}(\eta)\,d\eta. \tag{3.18}$$

With $s' = s/N$ and $u' = u/N$, we can write

$$\mathbb{E}_\omega\Big((\mathbf{p}_N(s)\psi, \mathbf{p}_N(u)\psi)\Big) = \lim_{\epsilon\downarrow 0}\mathbb{E}_\omega\Big((\psi, [\mathbf{V}_\epsilon(s')\mathbf{W}(s')]^N\mathbf{p}[\mathbf{W}(-s')\mathbf{V}_\epsilon(-s')]^N$$

$$\times [\mathbf{V}_\epsilon(u')\mathbf{W}(u')]^N\mathbf{p}[\mathbf{W}(-u')\mathbf{V}_\epsilon(-u')]^N\psi)\Big). \tag{3.19}$$

Remark 9

(i) Alternatively, one can first combine the two unitary groups $\mathbf{U}^*(s)\mathbf{U}(u)$ in the middle of the product $\mathbf{p}(s)\cdot\mathbf{p}(u)$ into just one unitary group, e.g. $\mathbf{U}(u-s)$, then apply the Trotter product theorem to the latter. This approach has an advantage that there are fewer convolutions when we apply the Fourier transform; see (3.20). However, it breaks the symmetry of s and u, giving rise to a more difficult combinatoric structure, especially for large N, when we take the expectation $\mathbb{E}_\omega(\cdot)$. See Sect. 3.4.

(ii) One may apply the Trotter product theorem directly to $\mathbf{X}(t)$ given by (2.5) in the position space. This leads one to study the sum of products of Bessel functions of varying orders (which depend on the distance between two lattice points at which the potential terms $\mathbf{V}(t)$ are evaluated). We will see that, in our current setting, the approximated expected value in (3.19) can be expressed in terms of oscillatory integrals, which indeed are equivalent to the sum of the products of the Bessel functions, but appear easier to estimate than their counterparts in the position space. Moreover, the Trotter product approximations directly to $\mathbf{X}(t)$ for $N = 1$ and $N = 2$ yield $\mathbb{E}_\omega \|\mathbf{X}_N(t)\delta_{\mathbf{0}}\|^2$ grows ballistically like t^2, in contrast with those for $\mathbf{p}(t)$; see also Sect. 3.7.

Hereafter, we will compute the approximated expectation given by (3.19). To do that, we need to isolate the potential terms from the free energy terms.

3.3 Expected Value in Momentum Space

Consider the approximated expected value $\mathbb{E}_\omega((\psi, \mathbf{p}_N(s)\cdot \mathbf{p}_N(u)\psi))$ in (3.19). Using convolution, the definition of $V_t(\mathbf{x})$ in (3.17), and that the Fourier transform is an isometry on $\ell^2(\mathbb{Z}^d)$, we derive the following expressions.

Lemma 3 *Let φ, ψ denote vectors in $\ell^2(\mathbb{Z}^d)$, and let $\kappa_j \in \mathbb{T}^d$, for $j = 0, 1, 2, \ldots$.*

(i) *For any integer $M \geq 1$,*

$$\begin{aligned}
&\mathfrak{F}\Big([\mathbf{V}_\epsilon(s)\mathbf{W}(s)]^M\varphi\Big)(\kappa_m)\\
&\quad= \int_{\mathbb{T}^d} \hat{V}_s(\kappa_m-\kappa_{m+1})G_s(\kappa_{m+1})\mathfrak{F}\Big([\mathbf{V}_\epsilon(s)\mathbf{W}(s)]^{M-1}\varphi\Big)(\kappa_{m+1})\,d\kappa_{m+1}\\
&\quad= \int_{(\mathbb{T}^d)^M}\left[\prod_{j=m+1}^{m+M} G_s(\kappa_j)\hat{V}_s(\kappa_{j-1}-\kappa_j)\right](\mathfrak{F}\varphi)(\kappa_{m+M})\prod_{j=m+1}^{m+M} d\kappa_j.
\end{aligned}$$

(ii) *For any integer $N \geq 1$,*

$$\begin{aligned}
&(\psi, \mathbf{p}_N(s)\cdot\mathbf{p}_N(u)\psi)\\
&\quad= (\psi, [\mathbf{V}_\epsilon(s')\mathbf{W}(s')]^N\mathbf{p}[\mathbf{W}(-s')\mathbf{V}_\epsilon(-s')]^N\\
&\qquad\times[\mathbf{V}_\epsilon(u')\mathbf{W}(u')]^N\mathbf{p}[\mathbf{W}(-u')\mathbf{V}_\epsilon(-u')]^N\psi)\\
&\quad= \int_{(\mathbb{T}^d)^{4N+1}}\left[\prod_{\substack{j=1\\ j\neq N,2N,3N}}^{4N-1} G_{t_j}(\kappa_j)\right]\mathfrak{p}(\kappa_N)\cdot\mathfrak{p}(\kappa_{3N})\left[\prod_{j=1}^{4N}\hat{V}_{t_j}(\kappa_{j-1}-\kappa_j)\right]\\
&\qquad\times\overline{\hat{\psi}(\kappa_0)}\hat{\psi}(\kappa_{4N})\prod_{j=0}^{4N} d\kappa_j,
\end{aligned}\tag{3.20}$$

where

$$t_j = \begin{cases} s' & \text{if } 1\leq j\leq N,\\ -s' & \text{if } N+1\leq j\leq 2N,\\ u' & \text{if } 2N+1\leq j\leq 3N,\\ -u' & \text{if } 3N+1\leq j\leq 4N.\end{cases}\tag{3.21}$$

Proof The identity (i) follows from the definition of convolution and an induction argument. By applying (i), we derive (ii) below.

$$
\begin{aligned}
&(\psi, \mathbf{p}_N(s)\cdot\mathbf{p}_N(u)\psi)\\
&\quad = (\psi, [\mathbf{V}_\epsilon(s')\mathbf{W}(s')]^N\mathbf{p}[\mathbf{W}(-s')\mathbf{V}_\epsilon(-s')]^N\cdot[\mathbf{V}_\epsilon(u')\mathbf{W}(u')]^N\mathbf{p}[\mathbf{W}(-u')\mathbf{V}_\epsilon(-u')]^N\psi)\\
&\quad = \int_{\mathbb{T}^d}\overline{\hat{\psi}(\kappa_0)}\mathfrak{F}\Big[[\mathbf{V}_\epsilon(s')\mathbf{W}(s')]^N\mathbf{p}[\mathbf{W}(-s')\mathbf{V}_\epsilon(-s')]^N\\
&\qquad\times[\mathbf{V}_\epsilon(u')\mathbf{W}(u')]^N\mathbf{p}[\mathbf{W}(-u')\mathbf{V}_\epsilon(-u')]^N\psi\Big](\kappa_0)\,d\kappa_0\\
&\quad = \int_{(\mathbb{T}^d)^{N+1}}\overline{\hat{\psi}(\kappa_0)}\prod_{j=1}^{N}\Big[G_{s'}(\kappa_j)\hat{V}_{s'}(\kappa_{j-1}-\kappa_j)\Big]\mathfrak{F}\Big[\mathbf{p}[\mathbf{W}(-s')\mathbf{V}_\epsilon(-s')]^N\\
&\qquad\times[\mathbf{V}_\epsilon(u')\mathbf{W}(u')]^N\mathbf{p}[\mathbf{W}(-u')\mathbf{V}_\epsilon(-u')]^N\psi\Big](\kappa_N)\prod_{j=0}^{N}d\kappa_j\\
&\quad = \int_{(\mathbb{T}^d)^{N+1}}\overline{\hat{\psi}(\kappa_0)}\prod_{j=1}^{N}\Big[G_{s'}(\kappa_j)\hat{V}_{s'}(\kappa_{j-1}-\kappa_j)\Big]\mathfrak{p}(\kappa_N)G_{-s'}(\kappa_N)\mathfrak{F}\Big[[\mathbf{V}_\epsilon(-s')\mathbf{W}(-s')]^{N-1}\\
&\qquad\times\mathbf{V}_\epsilon(-s')\cdot[\mathbf{V}_\epsilon(u')\mathbf{W}(u')]^N\mathbf{p}[\mathbf{W}(-u')\mathbf{V}_\epsilon(-u')]^N\psi\Big](\kappa_N)\prod_{j=0}^{N}d\kappa_j\\
&\quad = \int_{(\mathbb{T}^d)^{2N}}\overline{\hat{\psi}(\kappa_0)}\prod_{j=1}^{N}\Big[G_{s'}(\kappa_j)\hat{V}_{s'}(\kappa_{j-1}-\kappa_j)\Big]\mathfrak{p}(\kappa_N)G_{-s'}(\kappa_N)\\
&\qquad\times\prod_{j=N+1}^{2N-1}\Big[G_{-s'}(\kappa_j)\hat{V}_{-s'}(\kappa_{j-1}-\kappa_j)\Big]\\
&\qquad\times\mathfrak{F}\Big[\mathbf{V}_\epsilon(-s')\cdot[\mathbf{V}_\epsilon(u')\mathbf{W}(u')]^N\mathbf{p}[\mathbf{W}(-u')\mathbf{V}_\epsilon(-u')]^N\psi\Big](\kappa_{2N-1})\prod_{j=0}^{2N-1}d\kappa_j\\
&\quad = \int_{(\mathbb{T}^d)^{2N+1}}\overline{\hat{\psi}(\kappa_0)}\prod_{j=1}^{N}\Big[G_{s'}(\kappa_j)\hat{V}_{s'}(\kappa_{j-1}-\kappa_j)\Big]\mathfrak{p}(\kappa_N)G_{-s'}(\kappa_N)\\
&\qquad\times\prod_{j=N+1}^{2N-1}\Big[G_{-s'}(\kappa_j)\hat{V}_{-s'}(\kappa_{j-1}-\kappa_j)\Big]\\
&\qquad\times\hat{V}_{-s'}(\kappa_{2N-1}-\kappa_{2N})\cdot\mathfrak{F}\Big[[\mathbf{V}_\epsilon(u')\mathbf{W}(u')]^N\mathbf{p}[\mathbf{W}(-u')\mathbf{V}_\epsilon(-u')]^N\psi\Big](\kappa_{2N})\prod_{j=0}^{2N}d\kappa_j\\
&\quad = \int_{(\mathbb{T}^d)^{2N+1}}\overline{\hat{\psi}(\kappa_0)}\prod_{j=1}^{N-1}G_{s'}(\kappa_j)\prod_{j=N+1}^{2N-1}G_{-s'}(\kappa_j)\prod_{j=1}^{N}\hat{V}_{s'}(\kappa_{j-1}-\kappa_j)\prod_{j=N+1}^{2N}\hat{V}_{-s'}(\kappa_{j-1}-\kappa_j)\mathfrak{p}(\kappa_N)
\end{aligned}
$$

$$\times\, \mathfrak{F}\Big[[\mathbf{V}_\epsilon(u')\mathbf{W}(u')]^N \mathbf{p}[\mathbf{W}(-u')\mathbf{V}_\epsilon(-u')]^N \psi\Big](\kappa_{2N}) \prod_{j=0}^{2N} d\kappa_j$$

$$= \int_{(\mathbb{T}^d)^{4N+1}} \overline{\hat{\psi}(\kappa_0)} \prod_{j=1}^{N-1} G_{s'}(\kappa_j) \prod_{j=N+1}^{2N-1} G_{-s'}(\kappa_j) \prod_{j=1}^{N} \hat{V}_{s'}(\kappa_{j-1}-\kappa_j)$$

$$\times \prod_{j=N+1}^{2N} \hat{V}_{-s'}(\kappa_{j-1}-\kappa_j)\mathfrak{p}(\kappa_N) \prod_{j=2N+1}^{3N-1} G_{u'}(\kappa_j) \prod_{j=2N+1}^{3N} \hat{V}_{u'}(\kappa_j)\mathfrak{p}(\kappa_{3N}) \prod_{3N+1}^{4N-1} G_{-u'}(\kappa_j)$$

$$\times \prod_{j=3N+1}^{4N} \hat{V}_{-u'}(\kappa_j)\hat{\psi}(\kappa_{4N}) \prod_{j=0}^{4N} d\kappa_j.$$

The resulting equation (3.20) is obtained by collecting the potential terms and the free energy terms, while noting the definition of t_j. □

We note that $G_{s'}(\kappa_N), G_{-s'}(\kappa_{2N}), G_{u'}(\kappa_{3N})$ and $G_{-u'}(\kappa_{4N})$ are absent on the right-hand side of (3.20) because $G_{s'}(\kappa_N)$ and $G_{-s'}(\kappa_N)$ cancel each other, as do $G_{u'}(\kappa_{3N})$ and $G_{-u'}(\kappa_{3N})$, while $G_{-s'}(\kappa_{2N})$ and $G_{-u'}(\kappa_{4N})$ never appear. For compact notation, we define

$$\Psi(\kappa_N, \kappa_{3N}) := \mathfrak{p}(\kappa_N)\cdot\mathfrak{p}(\kappa_{3N}) = 4\sum_{l=1}^{d} \sin(\kappa_{N,l})\sin(\kappa_{3N,l}), \tag{3.22}$$

where $\kappa_{N,l}$ is the l-th component of $\kappa_N \in \mathbb{T}^d$. Moreover, let

$$\mathcal{F}(N,\boldsymbol{\kappa}) = \frac{1}{N}\left(\sum_{j=1}^{N-1}\xi(\kappa_j) - \sum_{j=N+1}^{2N-1}\xi(\kappa_j)\right), \tag{3.23}$$

$$\mathcal{G}(N,\boldsymbol{\kappa}) = \frac{1}{N}\left(\sum_{j=2N+1}^{3N-1}\xi(\kappa_j) - \sum_{j=3N+1}^{4N-1}\xi(\kappa_j)\right), \tag{3.24}$$

with $\boldsymbol{\kappa} = (\kappa_0, \kappa_1, \ldots, \kappa_{4N})$, where $\kappa_j \in \mathbb{T}^d$ for each $j = 0, 1, \ldots, 4N$. Then,

$$\prod_{\substack{j=1\\ j\neq N,2N,3N}}^{4N-1} G_{t_j}(\kappa_j) = e^{is\mathcal{F}(N,\boldsymbol{\kappa})} e^{iu\mathcal{G}(N,\boldsymbol{\kappa})}. \tag{3.25}$$

Whether or not Anderson localization is exhibited depends on the choice of the initial state ψ and the strength of disorder, as earlier indicated. It should be emphasized that, even at weak disorder, if ψ lies in the spectral subspace associated with band-edges, then Anderson localization has been proved in the spectral approach [2, 7, 15, 16, 18, 22, 27, 29, 36]. For large disorder or energies near the band edge, it has been shown that $r_\psi^2(t)$ is uniformly bounded in t with probability one [15, 16, 28]. Generally, (1.1) is studied when the initial wave function ψ is well localized in the space and energy support. In this work, we consider the initial state ψ that is spectrally spread out and would like to capture the fast component of $r_\psi(t)$. For our purpose, we will take $\psi = \delta_{\mathbf{0}}$. With the application of Lemma 3 and Fubini theorem, we obtain

$$\mathbb{E}_\omega(\delta_{\mathbf{0}}, \mathbf{p}_N(s)\cdot\mathbf{p}_N(u)\delta_{\mathbf{0}}) = \lim_{\epsilon\downarrow 0}\int_{(\mathbb{T}^d)^{4N+1}} e^{is\mathcal{F}(N,\boldsymbol{\kappa})}e^{iu\mathcal{G}(N,\boldsymbol{\kappa})}\Psi(\kappa_N,\kappa_{3N}) \times \mathbb{E}_\omega\left[\prod_{j=1}^{4N}\hat{V}_{t_j}(\kappa_{j-1}-\kappa_j)\right] d^{4N+1}\boldsymbol{\kappa}. \tag{3.26}$$

As a result, we obtain

$$r^2_{N,\delta_{\mathbf{0}}}(t) := \mathbb{E}_\omega\|\mathbf{X}_N(t)\delta_{\mathbf{0}}\|^2 = \int_0^t\int_0^t \lim_{\epsilon\downarrow 0}\int_{(\mathbb{T}^d)^{4N+1}} e^{is\mathcal{F}(N,\boldsymbol{\kappa})}e^{iu\mathcal{G}(N,\boldsymbol{\kappa})}\Psi(\kappa_N,\kappa_{3N}) \times \mathbb{E}_\omega\left[\prod_{j=1}^{4N}\hat{V}_{t_j}(\kappa_{j-1}-\kappa_j)\right] d^{4N+1}\boldsymbol{\kappa}\, ds du. \tag{3.27}$$

If $\mathbf{V}\equiv 0$, then by (3.17), $V_t(\mathbf{x}) = e^{-\epsilon\|\mathbf{x}\|}$ independent of t. Thus, by Lemma 17,

$$\begin{aligned}\|\mathbf{X}_N(t)\delta_{\mathbf{0}}\|^2 &= \int_0^t\int_0^t\int_{(\mathbb{T}^d)^{4N+1}} e^{is\mathcal{F}(N,\boldsymbol{\kappa})}e^{iu\mathcal{G}(N,\boldsymbol{\kappa})}\Psi(\kappa_N,\kappa_{3N}) \\ &\quad\times\prod_{j=1}^{4N}\delta(\kappa_{j-1}-\kappa_j)\, d^{4N+1}\boldsymbol{\kappa}\, dsdu \\ &= \int_0^t\int_0^t\int_{\mathbb{T}^d}\Psi(\kappa_0,\kappa_0)\, d\kappa_0 dsdu \\ &= \int_0^t\int_0^t\int_{\mathbb{T}^d} 4\sum_{j=1}^d \sin^2(\kappa_{0,j})\prod_{j=1}^d d\kappa_{0,j} \\ &= (2d)t^2 := \|\mathbf{p}\delta_0\|^2 t^2,\end{aligned}$$

independent of N. The same statement holds when $\mathbf{V}$ is any constant multiplication operator, consistent with the fact that, under this assumption, $\mathbf{H}$ and $\mathbf{p}$ commute, so $\|\mathbf{X}(t)\delta_{\mathbf{0}}\|^2 = \|\mathbf{p}\delta_0\|^2 t^2$ trivially. In general, we must compute $\mathbb{E}_\omega[\prod_{j=1}^{4N}\hat{V}_{t_j}(\kappa_{j-1}-\kappa_j)]$. This task is complicated by the fact that the terms in the product need not be independent. A basic idea is to write the above product as the sum of products of independent potential terms. We follow the approach of [8] and use the combinatoric structure described below to accomplish this task.

3.4 Combinatorics

For a fixed but arbitrary $N<\infty$, let

$$\mathfrak{S} = \Big\{1, 2, \ldots, 4N-1, 4N\Big\}.$$

Definition 5 A partition π of the set $\mathfrak{S}$ is a family of pairwise disjoint nonempty sets, denoted by $\pi = \{S_1, S_2, \ldots, S_m\}$, such that $\bigcup_{l=1}^m S_l = \mathfrak{S}$.

Definition 6 As in [8], $\pi = \{S_1, S_2, \dots, S_m\}$ is called a partition of *length* m, and a subset S_l is called a *block* of π. The *length* of π is often denoted by $|\pi|$, and the *size*, i.e. cardinality, of a block S_l is denoted by $|S_l|$.

It is important that we consider only distinct partitions. To do this, we always order the blocks S_l by their minimal elements $\sigma(l) := \min\{q : q \in S_l\}$. For our purpose, we can also order the elements in each block. For examples, $\pi = \{\{1, 2, 3, 4\}\}$ or $\pi = \{\{1, 4\}, \{2, 3\}\}$ when $N = 1$.

Definition 7 Two partitions $\pi = \{S_1, S_2, \dots, S_m\}$ and $\pi' = \{S'_1, S'_2, \dots, S'_{m'}\}$ are the same if $m = m'$ and $S_j = S'_j$ for each $1 \le j \le m$.

Definition 8 Let $\mathcal{P}$ be the collection of distinct partitions of the set $\mathfrak{S}$.

Definition 9 A partition $\pi' = \{S'_1, \dots, S'_{m'}\}$ is a refinement of a partition $\pi = \{S_1, S_2, \dots, S_m\}$ if (i) $m' > m$, and (ii) each S'_j, for $1 \le j \le m'$, is a subset of some S_l, for $1 \le l \le m$. In this case, we say that π' is a *subpartition* of π and denote this relationship by $\pi' \prec \pi$. Equivalently, we say that π is a *superpartition* of π', which will be denoted by $\pi \succ \pi'$.

Example 3.1 For $N = 2$, $\pi_1 = \{\{1, 2\}, \{3, 4\}, \{5, 6, 7, 8\}\}$ is a subpartition of $\pi = \{\{1, 2, 3, 4\}, \{5, 6, 7, 8\}\}$, while $\pi_2 = \{\{1, 5\}, \{2, 6, 7, 8\}, \{3, 4\}\}$ is not.

Notation 1 *We adopt the following notations*:

$$\begin{aligned}
\zeta_j &:= \kappa_{j-1} - \kappa_j \quad \textit{for } j = 0, \dots, 4N, \textit{ with } \kappa_{-1} = 0, \\
n_S &:= \{\mathbf{n}_j : j \in S\}, \\
\delta(n_S) &:= \sum_{\mathbf{x} \in \mathbb{Z}^d} \prod_{j \in S} \delta_{\mathbf{x}, \mathbf{n}_j} \quad \textit{for each } S \subset \mathfrak{S}, \\
\check{\mu}(S) &:= \check{\mu}\Big(\sum_{l \in S} t_l\Big) \quad \textit{for each } S \subset \mathfrak{S}, \textit{ where } t_l \textit{ is given by } (3.21), \\
\check{\mu}_\pi(s, u) &:= \prod_{l=1}^{m} \check{\mu}(S_l) \quad \textit{for } \pi = \{S_1, S_2, \dots, S_m\}.
\end{aligned}$$

In words, $\delta(n_S)$ is equal to one if all $\mathbf{n}_j$'s are equal for all $j \in S$ and zero otherwise. To be precise with the subsets of π, we write $\pi := \{S_1(\pi), S_2(\pi), \dots, S_{|\pi|}(\pi)\}$. With the above notations, we can write

$$\kappa_j = -\sum_{l=0}^{j} \zeta_l, \tag{3.28}$$

$$\mathbb{E}_\omega\left[\prod_{j=1}^{4N} \hat{V}_{t_j}(\kappa_{j-1} - \kappa_j)\right] = \mathbb{E}_\omega\left[\prod_{j=1}^{4N} \hat{V}_{t_j}(\zeta_j)\right]. \tag{3.29}$$

Lemma 4 *For each* $(\mathbf{n}_1, \mathbf{n}_2, \dots, \mathbf{n}_{4N}) \in (\mathbb{Z}^d)^{4N}$,

$$1 = \sum_{\substack{\pi \in \mathcal{P} \\ \pi = \{S_1(\pi), \dots, S_{|\pi|}(\pi)\}}} \prod_{l=1}^{|\pi|} \delta(n_{S_l(\pi)}) \prod_{1 \le i < j \le |\pi|} (1 - \delta_{n_{S_i(\pi)}, n_{S_j(\pi)}}), \tag{3.30}$$

where $S_i(\pi)$ *and* $S_j(\pi)$ *are disjoint if* $i \neq j$.

Proof Let $\mathbf{n}_1, \mathbf{n}_2, \dots, \mathbf{n}_{4N}$ denote $4N$ lattice points in $\mathbb{Z}^d$ which can be the same or different, and let $\mathfrak{S} = \{1, \dots, 4N\}$. We observe that each term in the sum on the right-hand side of (3.30) is either zero or one. It suffices to prove that there is exactly one partition associated with which the two products are both equal to one. To that end, we remark that such a partition π is formed by putting all the indices (i.e., elements in $\mathfrak{S}$) which correspond to the same lattice point into the same set. □

Lemma 5

$$\mathbb{E}_\omega \left\{ \prod_{j=1}^{4N} \hat{V}_{t_j}(\zeta_j) \right\} = \sum_{\mathbf{n}_1} \cdots \sum_{\mathbf{n}_{4N}} \left[\prod_{j=1}^{4N} e^{-i\zeta_j \cdot \mathbf{n}_j} e^{-\epsilon \|\mathbf{n}_j\|} \right] \sum_{\substack{\pi \in \mathcal{P} \\ \pi = \{S_1(\pi), \dots, S_{|\pi|}(\pi)\}}} \check{\mu}_\pi(s, u)$$
$$\times \prod_{l=1}^{|\pi|} \delta(n_{S_l(\pi)}) \prod_{1 \le i < j \le |\pi|} (1 - \delta_{n_{S_i(\pi)}, n_{S_j(\pi)}}). \tag{3.31}$$

Proof Using (3.30),

$$\mathbb{E}_\omega \left\{ \prod_{j=1}^{4N} \hat{V}_{t_j}(\zeta_j) \right\} = \mathbb{E}_\omega \left\{ \underbrace{\sum_{\mathbf{n}_1} \cdots \sum_{\mathbf{n}_{4N}}}_{4N \text{ sums}} \left[\prod_{j=1}^{4N} e^{-i\zeta_j \cdot \mathbf{n}_j} e^{it_j v_\omega(\mathbf{n}_j)} e^{-\epsilon \|\mathbf{n}_j\|} \right] \right\}$$
$$= \sum_{\mathbf{n}_1} \cdots \sum_{\mathbf{n}_{4N}} \left[\prod_{j=1}^{4N} e^{-i\zeta_j \cdot \mathbf{n}_j} e^{-\epsilon \|\mathbf{n}_j\|} \right] \mathbb{E}_\omega \left[\prod_{j=1}^{4N} e^{it_j v_\omega(\mathbf{n}_j)} \right]$$
$$= \sum_{\mathbf{n}_1} \cdots \sum_{\mathbf{n}_{4N}} \left[\prod_{j=1}^{4N} e^{-i\zeta_j \cdot \mathbf{n}_j} e^{-\epsilon \|\mathbf{n}_j\|} \right] \mathbb{E}_\omega \left[\prod_{j=1}^{4N} e^{it_j v_\omega(\mathbf{n}_j)} \right.$$
$$\left. \times \sum_{\substack{\pi \in \mathcal{P} \\ \pi = \{S_1(\pi), \dots, S_{|\pi|}(\pi)\}}} \prod_{l=1}^{|\pi|} \delta(n_{S_l(\pi)}) \prod_{1 \le i < j \le |\pi|} (1 - \delta_{n_{S_i(\pi)}, n_{S_j(\pi)}}) \right]$$
$$= \sum_{\mathbf{n}_1} \cdots \sum_{\mathbf{n}_{4N}} \left[\prod_{j=1}^{4N} e^{-i\zeta_j \cdot \mathbf{n}_j} e^{-\epsilon \|\mathbf{n}_j\|} \right]$$
$$\times \sum_{\substack{\pi \in \mathcal{P} \\ \pi = \{S_1(\pi), \dots, S_{|\pi|}(\pi)\}}} \mathbb{E}_\omega \left[\prod_{k=1}^{|\pi|} e^{i(\sum_{l \in S_k(\pi)} t_l) v_\omega(\mathbf{n}_{\sigma(l)})} \prod_{k=1}^{|\pi|} \delta(n_{S_k(\pi)}) \right)$$

$$\times \prod_{1\le i<j\le|\pi|}(1-\delta_{n_{S_i(\pi)},n_{S_j(\pi)}})\Bigg]$$

$$=\sum_{\mathbf{n}_1}\cdots\sum_{\mathbf{n}_{4N}}\left[\prod_{j=1}^{4N}e^{-i\zeta_j\cdot\mathbf{n}_j}e^{-\epsilon\|\mathbf{n}_j\|}\right]\Bigg\{\sum_{\substack{\pi\in\mathcal{P}\\ \pi=\{S_1(\pi),\ldots,S_{|\pi|}(\pi)\}}}\prod_{k=1}^{|\pi|}\check{\mu}\big(S_k(\pi)\big)$$

$$\times\prod_{k=1}^{|\pi|}\delta(n_{S_k(\pi)})\prod_{1\le i<j\le|\pi|}(1-\delta_{n_{S_i(\pi)},n_{S_j(\pi)}})\Bigg\},$$

where the last equality follows from (3.1). □

Example 3.2 If all points in the potential terms are the same, i.e., $\pi=\{\{1,2,3,\ldots,4N\}\}:=\{S_1(\pi)\}$, then by the definition of t_j in (3.21),

$$\check{\mu}_\pi(s,u)=\check{\mu}(S_1(\pi))=\check{\mu}\left(\sum_{j=1}^{4N}t_j\right)=\check{\mu}(0)\equiv 1.$$

In this case, we do not get any decay from the potential terms. In particular, the contribution from this partition to $\mathbb{E}_\omega\|\mathbf{X}_N(t)\delta_{\mathbf{0}}\|^2$ (see (3.23), (3.24), and (3.27)) is given by

$$\int_0^t\int_0^t\lim_{\epsilon\downarrow 0}\int_{(\mathbb{T}^d)^{4N+1}}e^{is\mathcal{F}(N,\boldsymbol{\kappa})}e^{iu\mathcal{G}(N,\boldsymbol{\kappa})}\Psi(\kappa_N,\kappa_{3N})$$

$$\times\sum_{\mathbf{n}_1,\ldots,\mathbf{n}_{4N}}\prod_{j=1}^{4N}e^{-i\zeta_j\cdot\mathbf{n}_j}e^{-\epsilon\|\mathbf{n}_j\|}\delta(n_{S_1(\pi)})\,d^{4N+1}\boldsymbol{\kappa}\,ds\,du$$

$$=\int_0^t\int_0^t\lim_{\epsilon\downarrow 0}\int_{(\mathbb{T}^d)^{4N+1}}e^{is\mathcal{F}(N,\boldsymbol{\kappa})}e^{iu\mathcal{G}(N,\boldsymbol{\kappa})}\Psi(\kappa_N,\kappa_{3N})$$

$$\times\sum_{\mathbf{n}}e^{-i(\zeta_1+\ldots+\zeta_{4N})\cdot\mathbf{n}}e^{-4N\epsilon\|\mathbf{n}\|}\,d^{4N+1}\boldsymbol{\kappa}\,ds\,du$$

$$=\int_0^t\int_0^t\int_{(\mathbb{T}^d)^{4N+1}}e^{is\mathcal{F}(N,\boldsymbol{\kappa})}e^{iu\mathcal{G}(N,\boldsymbol{\kappa})}\Psi(\kappa_N,\kappa_{3N})\delta(\zeta_1+\ldots+\zeta_{4N})\,d^{4N+1}\boldsymbol{\kappa}\,ds\,du$$

$$=\int_0^t\int_0^t\int_{(\mathbb{T}^d)^{4N+1}}e^{is\mathcal{F}(N,\boldsymbol{\kappa})}e^{iu\mathcal{G}(N,\boldsymbol{\kappa})}\Psi(\kappa_N,\kappa_{3N})\delta(\kappa_0-\kappa_{4N})\,d^{4N+1}\boldsymbol{\kappa}\,ds\,du,$$

by Lemma 17. Since $\Psi(\kappa_N,\kappa_{3N})$ in (3.22) is an odd function of κ_N and κ_{3N}, it follows that the last integral is identically zero; hence, this partition does not make any contribution to $\mathbb{E}_\omega\|\mathbf{X}_N(t)\delta_{\mathbf{0}}\|^2$.

On the other extreme, if all of the potential points are distinct, i.e., $\pi=\{\{1\},\{2\},\ldots,\{4N\}\}$, then

$$\prod_{1\le i<j\le m}(1-\delta_{n_{S_i(\pi)},n_{S_j(\pi)}})=1.$$

Hence, by Lemma 17 and (3.31), a contribution from this partition to $\mathbb{E}_\omega\|\mathbf{X}_N(t)\delta_{\mathbf{0}}\|^2$ is

$$\int_0^t\int_0^t\int_{(\mathbb{T}^d)^{4N+1}} e^{is\mathcal{F}(N,\boldsymbol{\kappa})}e^{iu\mathcal{G}(N,\boldsymbol{\kappa})}\Psi(\kappa_N,\kappa_{3N})|\check{\mu}(s')|^{2N}|\check{\mu}(u')|^{2N}\prod_{j=1}^{4N}\delta(\kappa_{j-1}-\kappa_j)d^{4N+1}\boldsymbol{\kappa}\,dsdu,$$

where $\mathcal{F}(N,\boldsymbol{\kappa})$, $\mathcal{G}(N,\boldsymbol{\kappa})$ and $\Psi(\kappa_N,\kappa_{3N})$ are given in (3.23), (3.24) and (3.22), respectively. After integrating over the delta functions, the contribution is precisely

$$\left(\int_{\mathbb{T}^d}\Psi(\kappa_0,\kappa_0)\,d\kappa_0\right)\int_0^t\int_0^t|\check{\mu}(s')|^{2N}|\check{\mu}(u')|^{2N}\,dsdu=\|\mathbf{p}\delta_0\|^2N^2\left|\int_0^{t/N}|\check{\mu}(w)|^{2N}\,dw\right|^2.$$

From this we can explore assumptions on $\check{\mu}$ such that the last integral is at worst $O_N(\sqrt{t})$, for large t. In this example, $\hat{\mu}\in L^2(\mathbb{R})$ is sufficient.

3.5 Expansion

The finiteness of N allows us to expand the products of the delta functions on the right-hand side of (3.31). We follow the approach of [8], but need to keep track of $\check{\mu}_\pi(s,u)$. As in [8], we observe that

$$\delta(n_{S_i(\pi)})\delta(n_{S_j(\pi)})\delta_{n_{S_i(\pi)},n_{S_j(\pi)}}=\delta(n_{S_i(\pi)\cup S_j(\pi)}). \tag{3.32}$$

For this reason, we will call $\delta_{n_{S_i(\pi)},n_{S_j(\pi)}}$ the *link* of $S_i(\pi)$ and $S_j(\pi)$, and use it to expand

$$\prod_{l=1}^{|\pi|}\delta(n_{S_l(\pi)})\prod_{1\le i<j\le|\pi|}(1-\delta_{n_{S_i(\pi)},n_{S_j(\pi)}}). \tag{3.33}$$

Lemma 6

$$\sum_{\pi\in\mathcal{P}}\check{\mu}_\pi(s,u)\prod_{l=1}^{|\pi|}\delta(n_{S_l(\pi)})\prod_{1\le i<j\le|\pi|}(1-\delta_{n_{S_i(\pi)},n_{S_j(\pi)}})=\sum_{\pi\in\mathcal{P}}\check{\mathcal{U}}_\pi(s,u)\prod_{l=1}^{|\pi|}\delta(n_{S_l(\pi)}), \tag{3.34}$$

where

$$\check{\mathcal{U}}_\pi(s,u)=\sum_{\pi':\pi'\prec\pi}(-1)^{|\pi'|-|\pi|}n_{\pi,\pi'}\check{\mu}_{\pi'}(s,u), \tag{3.35}$$

for some positive integer $n_{\pi',\pi}$.

Proof For a fixed partition $\pi=\{S_1(\pi),S_2(\pi),\dots,S_m(\pi)\}\in\mathcal{P}$ with $|\pi|=m$, consider

$$\check{\mu}_\pi(s,u)\prod_{l=1}^{m}\delta(n_{S_l(\pi)})\prod_{1\le i<j\le m}(1-\delta_{n_{S_i(\pi)},n_{S_j(\pi)}})$$

on the left-hand side of (3.34). First, we will expand the second product of the delta functions. For this, we note that the subsets $S_l(\pi)$ (and their elements) are ordered so that

$$1 = n_{\sigma(1)} < n_{\sigma(2)} < \ldots < n_{\sigma(m)},$$

where $\sigma(i) := \min\{q : q \in S_i\}$. For $1 \le i < j \le m$, let $\epsilon_{ij} \in \{0, 1\}$, and let $r_{ij} := \delta_{n_{S_i(\pi)}, n_{S_j(\pi)}}$ denote a *link* between $S_i(\pi)$ and $S_j(\pi)$. Let $|\epsilon| := \sum_{1\le i<j\le m} \epsilon_{ij}$. Then,

$$\begin{aligned}\prod_{1\le i<j\le m} (1 - \delta_{n_{S_i(\pi)}, n_{S_j(\pi)}}) &:= \prod_{1\le i<j\le m} (1 - r_{ij}) \\ &= \sum_{\{\epsilon_{ij}\}} (-1)^{|\epsilon|} \prod_{1\le i<j\le m} r_{ij}^{\epsilon_{ij}},\end{aligned}$$

where the summation is taken over all possible configurations of $\{\epsilon_{ij}\}$. Note that there are $(m/2)(m-1)$ terms in the last product; hence, there are $2^{m(m-1)/2}$ possible configurations of $\{\epsilon_{ij}\}$. Moreover, each configuration determines a product of delta functions which carry out the instruction how to "join" the subsets using

$$\delta(n_{S_i(\pi)})\delta(n_{S_j(\pi)})\delta_{n_{S_i(\pi)}, n_{S_j(\pi)}} = \delta(n_{S_i(\pi)\cup S_j(\pi)}).$$

However, two products of delta functions can result in the same outcome. For instance, $r_{1,2}r_{1,3}$ and $r_{1,3}r_{2,3}$ both "combine" subsets $S_1(\pi)$, $S_2(\pi)$ and $S_3(\pi)$ into a bigger set $S_1(\pi) \cup S_2(\pi) \cup S_3(\pi)$. Then,

$$\begin{aligned}&\check{\mu}_\pi(s, u) \prod_{l=1}^{m} \delta(n_{S_l(\pi)}) \prod_{1\le i<j\le m} (1 - \delta_{n_{S_i(\pi)}, n_{S_j(\pi)}}) \\ &= \check{\mu}_\pi(s, u) \prod_{l=1}^{m} \delta(n_{S_l(\pi)}) \sum_{\{\epsilon_{ij}\}} (-1)^{|\epsilon|} \prod_{1\le i<j\le m} r_{ij}^{\epsilon_{ij}} \\ &= \check{\mu}_\pi(s, u) \prod_{l=1}^{m} \delta(n_{S_l(\pi)}) + \check{\mu}_\pi(s, u) \sum_{\{\epsilon_{ij}\}\setminus\{0,0,..,0\}} (-1)^{|\epsilon|} \prod_{l=1}^{m} \delta(n_{S_l(\pi)}) \prod_{1\le i<j\le m} r_{ij}^{\epsilon_{ij}},\end{aligned}$$

where the last summation does not include the configuration where all ϵ_{ij} are zero.

Next, if $\tilde{\pi}$ is a superpartition of π, then there exists at least one configuration of $\{\epsilon_{ij}\}$ which "connects" π and $\tilde{\pi}$. Also note that $|\epsilon|$ basically counts the number of *links* between π and $\tilde{\pi}$; thus, $|\epsilon| = |\pi| - |\tilde{\pi}|$. Therefore, the last sum can be expressed as

$$\check{\mu}_\pi(s, u) \sum_{\substack{\tilde{\pi}:\tilde{\pi}\succ\pi \\ \tilde{\pi}\ne\pi}} (-1)^{|\pi|-|\tilde{\pi}|} n_{\tilde{\pi},\pi} \prod_{l=1}^{|\tilde{\pi}|} \delta(n_{S_l(\tilde{\pi})}),$$

where $n_{\tilde{\pi},\pi}$ denotes the number of ways to "reconstruct" $\tilde{\pi}$ from π. As a result,

$$\check{\mu}_\pi(s, u) \prod_{l=1}^{m} \delta(n_{S_l(\pi)}) \prod_{1\le i<j\le m} (1 - \delta_{n_{S_i(\pi)}, n_{S_j(\pi)}}) = \check{\mu}_\pi(s, u) \sum_{\tilde{\pi}:\tilde{\pi}\succ\pi} (-1)^{|\pi|-|\tilde{\pi}|} n_{\tilde{\pi},\pi} \prod_{l=1}^{|\tilde{\pi}|} \delta(n_{S_l(\tilde{\pi})}). \tag{3.36}$$

Finally, the sum on the left-hand side of (3.34) can be written as

$$\begin{aligned}
&\sum_{\pi\in\mathcal{P}} \check{\mu}_\pi(s,u) \prod_{l=1}^{|\pi|} \delta(n_{S_l(\pi)}) \prod_{1\le i<j\le|\pi|} (1-\delta_{n_{S_i(\pi)},n_{S_j(\pi)}}) \\
&= \sum_{\pi\in\mathcal{P}} \check{\mu}_\pi(s,u) \sum_{\tilde{\pi}:\tilde{\pi}\succ\pi} (-1)^{|\pi|-|\tilde{\pi}|} n_{\tilde{\pi},\pi} \prod_{l=1}^{|\tilde{\pi}|} \delta(n_{S_l(\tilde{\pi})}) \\
&= \sum_{\tilde{\pi}\in\mathcal{P}} \left(\sum_{\pi:\pi\prec\tilde{\pi}} (-1)^{|\pi|-|\tilde{\pi}|} n_{\tilde{\pi},\pi} \check{\mu}_\pi(s,u) \right) \prod_{l=1}^{|\tilde{\pi}|} \delta(n_{S_l(\tilde{\pi})}) \\
&= \sum_{\tilde{\pi}\in\mathcal{P}} \check{\mathcal{U}}_{\tilde{\pi}}(s,u) \prod_{l=1}^{|\tilde{\pi}|} \delta(n_{S_l(\tilde{\pi})}) \\
&= \sum_{\pi\in\mathcal{P}} \check{\mathcal{U}}_\pi(s,u) \prod_{l=1}^{|\pi|} \delta(n_{S_l(\pi)}),
\end{aligned}$$

where $\check{\mathcal{U}}_\pi(s,u)$ is precisely as given in (3.35). □

Remark 10 We point out that, in (3.35), π may have a subpartition π' such that $\check{\mu}_{\pi'}(s,u) \equiv 1$. For example, $\pi = \{\{1,2,3,4\}\}$ and $\pi = \{\{1,2\},\{3,4\}\}$ when $N=1$. See Example 3.3, Lemma 11, and Corollary 2.

Corollary 1

$$\mathbb{E}_\omega \left\{ \prod_{j=1}^{4N} \hat{V}_{t_j}(\zeta_j) \right\} = \sum_{\mathbf{n}_1} \cdots \sum_{\mathbf{n}_{4N}} \left[\prod_{j=1}^{4N} e^{-i\zeta_j\cdot\mathbf{n}_j} e^{-\epsilon\|\mathbf{n}_j\|} \right] \times \sum_{\substack{\pi\in\mathcal{P} \\ \pi=\{S_1(\pi),\ldots,S_{|\pi|}(\pi)\}}} \check{\mathcal{U}}_\pi(s,u) \prod_{l=1}^{|\pi|} \delta(n_{S_l(\pi)}), \tag{3.37}$$

where $\check{\mathcal{U}}_\pi(s,u)$ *is given by* (3.35).

Proof Combine results from Lemmas 5 and 6. □

Lemma 7 *With* $\zeta_j = \kappa_{j-1} - \kappa_j \in \mathbb{T}^d$ *and* $\boldsymbol{\kappa} = (\kappa_0, \ldots, \kappa_{4N})$,

$$\mathbb{E}_\omega \|\mathbf{X}_N(t)\delta_{\mathbf{0}}\|^2 = \int_0^t \int_0^t \sum_{\pi\in\mathcal{P}} \check{\mathcal{U}}_\pi(s,u) \int_{(\mathbb{T}^d)^{4N+1}} e^{is\mathcal{F}(N,\boldsymbol{\kappa})} e^{iu\mathcal{G}(N,\boldsymbol{\kappa})} \Psi(\kappa_N,\kappa_{3N}) \times \prod_{l=1}^{|\pi|} \delta\left(\sum_{j\in S_l(\pi)} \zeta_j \right) d^{4N+1}\boldsymbol{\kappa}\, ds du. \tag{3.38}$$

Proof By (3.27) and Corollary 1,

$$
\begin{aligned}
&\mathbb{E}_\omega\|\mathbf{X}_N(t)\delta_{\mathbf{0}}\|^2 \\
&= \int_0^t\int_0^t \lim_{\epsilon\downarrow 0}\int_{(\mathbb{T}^d)^{4N+1}} e^{is\mathcal{F}(N,\boldsymbol{\kappa})}e^{iu\mathcal{G}(N,\boldsymbol{\kappa})}\Psi(\kappa_N,\kappa_{3N}) \\
&\quad\times \mathbb{E}_\omega\left[\prod_{j=1}^{4N}\hat{V}_{t_j}(\zeta_j)\right] d^{4N+1}\boldsymbol{\kappa}\, ds du \\
&= \int_0^t\int_0^t \sum_{\pi\in\mathcal{P}}\check{\mathcal{U}}_\pi(s,u)\lim_{\epsilon\downarrow 0}\int_{(\mathbb{T}^d)^{4N+1}} e^{is\mathcal{F}(N,\boldsymbol{\kappa})}e^{iu\mathcal{G}(N,\boldsymbol{\kappa})}\Psi(\kappa_N,\kappa_{3N}) \\
&\quad\times \sum_{\mathbf{n}_1}\cdots\sum_{\mathbf{n}_{4N}}\left[\prod_{j=1}^{4N}e^{-i\zeta_j\cdot\mathbf{n}_j}e^{-\epsilon\|\mathbf{n}_j\|}\prod_{l=1}^{|\pi|}\delta(n_{S_l(\pi)})\right] d^{4N+1}\boldsymbol{\kappa}\, ds du \\
&= \int_0^t\int_0^t \sum_{\pi\in\mathcal{P}}\check{\mathcal{U}}_\pi(s,u)\lim_{\epsilon\downarrow 0}\int_{(\mathbb{T}^d)^{4N+1}} e^{is\mathcal{F}(N,\boldsymbol{\kappa})}e^{iu\mathcal{G}(N,\boldsymbol{\kappa})}\Psi(\kappa_N,\kappa_{3N}) \\
&\quad\times \prod_{l=1}^{|\pi|}\Bigl(\sum_{\mathbf{n}_{\sigma(l)}}e^{-i\mathbf{n}_{\sigma(l)}\cdot(\sum_{j\in S_l(\pi)}\zeta_j)}e^{-\epsilon|S_j(\pi)|\|\mathbf{n}_{\sigma(l)}\|}\Bigr) d^{4N+1}\boldsymbol{\kappa}\, ds du \\
&= \int_0^t\int_0^t \sum_{\pi\in\mathcal{P}}\check{\mathcal{U}}_\pi(s,u)\int_{(\mathbb{T}^d)^{4N+1}} e^{is\mathcal{F}(N,\boldsymbol{\kappa})}e^{iu\mathcal{G}(N,\boldsymbol{\kappa})}\Psi(\kappa_N,\kappa_{3N}) \\
&\quad\times \prod_{l=1}^{|\pi|}\delta\Bigl(\sum_{j\in S_l(\pi)}\zeta_j\Bigr) d^{4N+1}\boldsymbol{\kappa}\, ds du,
\end{aligned}
$$

where the last equality holds by Lemma 17. □

Notation 2 *We introduce the following convenient notations*:

$$\boldsymbol{\zeta} := (\zeta_0,\zeta_1,\ldots,\zeta_{4N}) := (\zeta_0,\boldsymbol{\zeta}_\sharp), \tag{3.39}$$

$$\boldsymbol{\zeta}_i := (\zeta_{0,i},\zeta_{1,i},\ldots,\zeta_{4N,i}) := (\zeta_{0,i},\boldsymbol{\zeta}_{\sharp,i}), \quad \textit{for } 1\le i\le d, \tag{3.40}$$

$$K_\pi(\boldsymbol{\zeta}_\sharp) := \prod_{k=1}^{|\pi|}\delta\Bigl(\sum_{j\in S_k(\pi)}\zeta_j\Bigr), \tag{3.41}$$

$$\tilde{K}_\pi(\boldsymbol{\zeta}_{\sharp,i}) := \prod_{k=1}^{|\pi|}\delta\Bigl(\sum_{j\in S_k(\pi)}\zeta_{j,i}\Bigr), \tag{3.42}$$

where $\zeta_j\in\mathbb{T}^d$, for each $j=0,1,\ldots,4N$, and $\zeta_{j,l}$ is the l-component of ζ_j. Here, it should be cautioned that $\boldsymbol{\zeta}_l$ is not ζ_l. In fact, for each $l=1,\ldots,d$, $\boldsymbol{\zeta}_l\in\mathbb{T}^{4N+1}$, and its n-th component is $(\boldsymbol{\zeta}_l)_n=\zeta_{n,l}$ for $n=0,\ldots,4N$. The arguments of the delta functions in (3.41) are d-dimensional vectors, whereas those in (3.42) are one dimensional. Thus, $K_\pi(\boldsymbol{\zeta}_\sharp)$ and

$\tilde{K}_\pi(\boldsymbol{\zeta}_{\sharp,i})$ *are related by*

$$K_\pi(\boldsymbol{\zeta}_\sharp) = \prod_{i=1}^{d} \tilde{K}_\pi(\boldsymbol{\zeta}_{\sharp,i}).$$

When $d = 1$, *so that* $\zeta_j \in \mathbb{T}$ *for each* $0 \le j \le 4N$, *we put* $\zeta := \boldsymbol{\zeta}_1 = \boldsymbol{\zeta}$.

We refer to $K_\pi(\boldsymbol{\zeta}_\sharp)$ as the integration kernel associated with π. For each partition π, the arguments of the delta functions in $K_\pi(\boldsymbol{\zeta}_\sharp)$ define a homogeneous system of linear equations in which each ζ_j, for $1 \le i \le 4N$, appears exactly once with coefficient one, but ζ_0 never appears. Hence, each κ_j, for $1 \le j \le 4N - 1$, appears exactly twice with opposite signs, but κ_0 appears once with a plus sign, and κ_{4N} appears once with a minus sign. Though they are interchangeable, we will work with $\zeta_0, \dots, \zeta_{4N} \in \mathbb{T}^d$ as our variables rather than $\kappa_0, \dots, \kappa_{4N}$. For each π, there are $d|\pi|$ constraint equations which define a subspace in $(\mathbb{R}^d)^{4N+1}$. Because our underlying space is $(\mathbb{T}^d)^{4N+1}$ rather than $(\mathbb{R}^d)^{4N+1}$, we say that two sets of constraint equations are equivalent if they determine the same affine subspace of $(\mathbb{R}^d)^{4N+1}$ modulo 2π.

With notations in (3.28) and (3.39), we can rewrite $\mathcal{F}(N, \boldsymbol{\kappa})$ in (3.23) and $\mathcal{G}(N, \boldsymbol{\kappa})$ in (3.24) in terms of $\boldsymbol{\zeta}$ as

$$\tilde{\mathcal{F}}(N, \boldsymbol{\zeta}) = \frac{1}{N}\left(\sum_{j=1}^{N-1} \xi(\zeta_0 + \zeta_1 + \dots + \zeta_j) - \sum_{j=N+1}^{2N-1} \xi(\zeta_0 + \zeta_1 + \dots + \zeta_j)\right), \tag{3.43}$$

$$\tilde{\mathcal{G}}(N, \boldsymbol{\zeta}) = \frac{1}{N}\left(\sum_{j=2N+1}^{3N-1} \xi(\zeta_0 + \zeta_1 + \dots + \zeta_j) - \sum_{j=3N+1}^{4N-1} \xi(\zeta_0 + \zeta_1 + \dots + \zeta_j)\right). \tag{3.44}$$

Furthermore,

$$\begin{aligned}\Psi(\kappa_N, \kappa_{3N}) &= \Psi(-\zeta_0 - \dots - \zeta_N, -\zeta_0 - \dots - \zeta_{3N}) \\ &= \Psi(\zeta_0 + \dots + \zeta_N, \zeta_0 + \dots + \zeta_{3N}) \\ &= 4\sum_{j=1}^{d} \sin(\zeta_{0,j} + \dots + \zeta_{N,j}) \sin(\zeta_{0,j} + \dots + \zeta_{3N,j}).\end{aligned} \tag{3.45}$$

3.6 Equivalent Statements

Using (3.41) and (3.43)—(3.45), $\mathbb{E}_\omega \|\mathbf{X}_N(t)\delta_\mathbf{0}\|^2$ in Lemma 7 can be equivalently expressed as

$$\begin{aligned}\mathbb{E}_\omega \|\mathbf{X}_N(t)\delta_\mathbf{0}\|^2 = \sum_{\pi \in \mathcal{P}} \int_0^t \int_0^t \check{\mathcal{U}}_\pi(s, u) \int_{(\mathbb{T}^d)^{4N+1}} e^{is\tilde{\mathcal{F}}(N,\boldsymbol{\zeta})} e^{iu\tilde{\mathcal{G}}(N,\boldsymbol{\zeta})} \\ \times \Psi\big(\zeta_0 + \zeta_1 + \dots + \zeta_N, \zeta_0 + \zeta_1 + \dots + \zeta_{3N}\big) K_\pi(\boldsymbol{\zeta}_\sharp)\, d\boldsymbol{\zeta}\, ds du.\end{aligned} \tag{3.46}$$

From the definition of the free energy terms in Remark 6, both functions $\tilde{\mathcal{F}}(N, \boldsymbol{\zeta})$ and $\tilde{\mathcal{G}}(N, \boldsymbol{\zeta})$ can be separated into a sum of d terms, each of which corresponds to a one-dimensional problem. That is,

$$\tilde{\mathcal{F}}(N, \boldsymbol{\zeta}) = \sum_{l=1}^{d} f_l(N, \boldsymbol{\zeta}_l),$$

$$\tilde{\mathcal{G}}(N, \boldsymbol{\zeta}) = \sum_{l=1}^{d} g_l(N, \boldsymbol{\zeta}_l),$$

where, for each $l = 1, \ldots, d$,

$$f_l(N, \boldsymbol{\zeta}_l) = -\frac{2}{N}\left(\sum_{j=1}^{N-1}\cos(\zeta_{0,l} + \ldots + \zeta_{j,l}) - \sum_{j=N+1}^{2N-1}\cos(\zeta_{0,l} + \ldots + \zeta_{j,l})\right), \tag{3.47}$$

$$g_l(N, \boldsymbol{\zeta}_l) = -\frac{2}{N}\left(\sum_{j=2N+1}^{3N-1}\cos(\zeta_{0,l} + \ldots + \zeta_{j,l}) - \sum_{j=3N+1}^{4N-1}\cos(\zeta_{0,l} + \ldots + \zeta_{j,l})\right). \tag{3.48}$$

Combined with (3.42), this implies that

Lemma 8

$$\int_{(\mathbb{T}^d)^{4N+1}} e^{is\tilde{\mathcal{F}}(N,\boldsymbol{\zeta})}e^{iu\tilde{\mathcal{G}}(N,\boldsymbol{\zeta})}\Psi(\zeta_0 + \zeta_1 + \ldots + \zeta_N, \zeta_0 + \zeta_1 + \ldots + \zeta_{3N})K_\pi(\boldsymbol{\zeta}_\sharp)\,d\boldsymbol{\zeta}$$
$$= (4d)\ \Xi_\pi(s,u)\Big[\Lambda_\pi(s,u)\Big]^{d-1}, \tag{3.49}$$

where

$$\Xi_\pi(s,u) := \int_{\mathbb{T}^{4N+1}} e^{isf_1(N,\boldsymbol{\zeta}_1)}e^{iug_1(N,\boldsymbol{\zeta}_1)}\sin(\zeta_{0,1} + \zeta_{1,1} + \ldots + \zeta_{N,1})$$
$$\times \sin(\zeta_{0,1} + \zeta_{1,1} + \ldots + \zeta_{3N,1})\tilde{K}_\pi(\boldsymbol{\zeta}_{\sharp,1})\,d\boldsymbol{\zeta}_1, \tag{3.50}$$

and

$$\Lambda_\pi(s,u) := \int_{\mathbb{T}^{4N+1}} e^{isf_1(N,\boldsymbol{\zeta}_1)}e^{iug_1(N,\boldsymbol{\zeta}_1)}\tilde{K}_\pi(\boldsymbol{\zeta}_{\sharp,1})\,d\boldsymbol{\zeta}_1. \tag{3.51}$$

Proof

$$\int_{(\mathbb{T}^d)^{4N+1}} e^{is\tilde{\mathcal{F}}(N,\boldsymbol{\zeta})}e^{iu\tilde{\mathcal{G}}(N,\boldsymbol{\zeta})}\Psi(\zeta_0 + \zeta_1 + \ldots + \zeta_N, \zeta_0 + \zeta_1 + \ldots + \zeta_{3N})K_\pi(\boldsymbol{\zeta}_\sharp)\,d\boldsymbol{\zeta}\,dsdu$$
$$= \int_{(\mathbb{T}^d)^{4N+1}} e^{is\tilde{\mathcal{F}}(N,\boldsymbol{\zeta})}e^{iu\tilde{\mathcal{G}}(N,\boldsymbol{\zeta})}\Bigg[4\sum_{l=1}^{d}\sin(\zeta_{0,l} + \zeta_{1,l} + \ldots + \zeta_{N,l})$$
$$\times \sin(\zeta_{0,l} + \zeta_{1,l} + \ldots + \zeta_{3N,l})\Bigg]\prod_{l=1}^{d}\tilde{K}_\pi(\boldsymbol{\zeta}_{\sharp,l})\prod_{l=1}^{d}d\boldsymbol{\zeta}_l,$$
$$= 4\sum_{l=1}^{d}\int_{(\mathbb{T}^d)^{4N+1}}\prod_{j=1}^{d}\Big[e^{isf_j(N,\boldsymbol{\zeta}_j)}e^{iug_j(N,\boldsymbol{\zeta}_j)}\Big]\sin(\zeta_{0,l} + \zeta_{1,l} + \ldots + \zeta_{N,l})$$
$$\times \sin(\zeta_{0,l} + \zeta_{1,l} + \ldots + \zeta_{3N,l})\prod_{j=1}^{d}\tilde{K}_\pi(\boldsymbol{\zeta}_{\sharp,j})\prod_{j=1}^{d}d\boldsymbol{\zeta}_j$$
$$= 4\sum_{l=1}^{d}\Bigg(\int_{\mathbb{T}^{4N+1}} e^{isf_l(N,\zeta_l)}e^{iug_l(N,\zeta_l)}\sin(\zeta_{0,l} + \zeta_{1,l} + \ldots + \zeta_{N,l})$$
$$\times \sin(\zeta_{0,l} + \zeta_{1,l} + \ldots + \zeta_{3N,l})\tilde{K}_\pi(\boldsymbol{\zeta}_{\sharp,l})\,d\boldsymbol{\zeta}_l\Bigg)$$

$$\times \prod_{\substack{q=1 \\ q\neq l}}^{d} \left[\int_{\mathbb{T}^{4N+1}} e^{isf_q(N,\boldsymbol{\zeta}_q)} e^{iug_q(N,\boldsymbol{\zeta}_q)} \tilde{K}_\pi(\boldsymbol{\zeta}_{\sharp,q})\, d\boldsymbol{\zeta}_q \right]$$

$$:= 4d\, \Xi_\pi(s,u) \prod_{l=1}^{d-1} \Lambda_{\pi,l}(s,u),$$

where

$$\Xi_\pi(s,u) := \int_{\mathbb{T}^{4N+1}} e^{isf_1(N,\boldsymbol{\zeta}_1)} e^{iug_1(N,\boldsymbol{\zeta}_1)} \sin(\zeta_{0,1} + \zeta_{1,1} + \ldots + \zeta_{N,1}) \times \sin(\zeta_{0,1} + \zeta_{1,1} + \ldots + \zeta_{3N,1}) \tilde{K}_\pi(\boldsymbol{\zeta}_{\sharp,1})\, d\boldsymbol{\zeta}_1,$$

$$\Lambda_{\pi,l}(s,u) := \int_{\mathbb{T}^{4N+1}} e^{isf_l(N,\boldsymbol{\zeta}_l)} e^{iug_l(N,\boldsymbol{\zeta}_l)} \tilde{K}_\pi(\boldsymbol{\zeta}_{\sharp,l})\, d\boldsymbol{\zeta}_l.$$

Since all $f_l(N,\boldsymbol{\zeta}_l)$, respectively all $g_l(N,\boldsymbol{\zeta}_l)$, have the same structure, it follows that all $\Lambda_{\pi,l}(s,u)$ are identical, hence $\prod_{l=1}^{d-1} \Lambda_{\pi,i}(s,u) = \Lambda_\pi(s,u)^{d-1}$, which proves the lemma. □

Therefore, we can write $\mathbb{E}_\omega \|\mathbf{X}_N(t)\delta_{\mathbf{0}}\|^2$ in (3.46) differently but equivalently as

$$\mathbb{E}_\omega \|\mathbf{X}_N(t)\delta_{\mathbf{0}}\|^2 = (4d) \sum_{\pi \in \mathcal{P}} \int_0^t \int_0^t \check{\mathcal{U}}_\pi(s,u)\, \Xi_\pi(s,u) \Big[\Lambda_\pi(s,u) \Big]^{d-1} dsdu. \tag{3.52}$$

For each $\pi \in \mathcal{P}$, we define the amplitude $A_\pi(t)$ and the free decay $\Gamma_\pi(t)$ of the wave function corresponding to the partition π as

$$A_\pi(t) := \int_0^t \int_0^t \int_{(\mathbb{T}^d)^{4N+1}} \check{\mathcal{U}}_\pi(s,u)\, e^{is\tilde{\mathcal{F}}(N,\boldsymbol{\zeta})} e^{iu\tilde{\mathcal{G}}(N,\boldsymbol{\zeta})} \times \Psi(\zeta_0 + \zeta_1 + \ldots + \zeta_N, \zeta_0 + \zeta_1 + \ldots + \zeta_{3N}) K_\pi(\boldsymbol{\zeta}_\sharp)\, d\boldsymbol{\zeta}\, dsdu \tag{3.53}$$

$$= (4d) \int_0^t \int_0^t \check{\mathcal{U}}_\pi(s,u)\, \Xi_\pi(s,u) \Big[\Lambda_\pi(s,u) \Big]^{d-1} dsdu, \tag{3.54}$$

$$\Gamma_\pi(t) := \int_0^t \int_0^t \int_{(\mathbb{T}^d)^{4N+1}} e^{is\tilde{\mathcal{F}}(N,\boldsymbol{\zeta})} e^{iu\tilde{\mathcal{G}}(N,\boldsymbol{\zeta})} \times \Psi(\zeta_0 + \zeta_1 + \ldots + \zeta_N, \zeta_0 + \zeta_1 + \ldots + \zeta_{3N}) K_\pi(\boldsymbol{\zeta}_\sharp)\, d\boldsymbol{\zeta}\, dsdu \tag{3.55}$$

$$= (4d) \int_0^t \int_0^t \Xi_\pi(s,u) \Big[\Lambda_\pi(s,u) \Big]^{d-1} dsdu. \tag{3.56}$$

Therefore, since N is finite, the proof of Proposition 1 is reduced to showing that $|A_\pi(t)| = O_N(t)$ for each $\pi \in \mathcal{P}$ when $d \geq 3$ and $\check{\mu} \in L^2(\mathbb{R})$. This depends on the structure of π, i.e., the constraint equations are governed by $K_\pi(\boldsymbol{\zeta}_\sharp)$. $A_\pi(t)$ (resp. $\Gamma_\pi(t)$) is an oscillatory integral whose two phases are given as sums of *cosines*. Consequently, after $K_\pi(\boldsymbol{\zeta}_\sharp)$ is integrated out, it is very difficult to explicitly describe the set of critical points, from which one expects a main contribution to the leading term of $A_\pi(t)$ in (3.53) (resp. that of $\Gamma_\pi(t)$ in (3.55)). We will consider $A_\pi(t)$ and $\Gamma_\pi(t)$ in details in Sect. 4. We close this section with some examples of computations for small values of N.

3.7 Trotter Product Approximations to $\mathbf{p}(t)$ for $N = 1$ and $N = 2$

Example 3.3 For $N = 1$, there are 15 partitions in $\mathcal{P}$, see Table 1. For each $\pi \in \mathcal{P}$, define

$$c_\pi(t) := \int_0^t \int_0^t |\check{\mu}_\pi(s, u)|\, ds du.$$

Suppose $\check{\mu} \in L^2(\mathbb{R})$, then, for each $\pi \in \mathcal{P}$, $\check{\mu}_\pi(s, u)$ and the corresponding bound of $c_\pi(t)$ are listed in Table 1 below. When computing the upper bound for $c_\pi(t)$, it is useful to note that $\|\check{\mu}\|_\infty \le \check{\mu}(0) = 1$ and that, by the Schwarz inequality,

$$\int_0^a |\check{\mu}(x)|\, dx \le \sqrt{a}\|\check{\mu}\|_2, \tag{3.57}$$

for any $a \ge 0$.

From Table 1, we learn that if π is such that $\sum_{j \in S_l(\pi)} t_j \not\equiv 0$ for some block $S_l(\pi)$ of π, i.e., $\check{\mu}_\pi(s, u) \not\equiv 1$, then $|c_\pi(t)| = O(t)$ as long as $\check{\mu} \in L^2(\mathbb{R})$. On the other hand, if $\check{\mu}(s, u) \equiv 1$, then we need to show that the oscillatory integral

$$\Gamma_\pi(t) = (4d) \int_0^t \int_0^t \Xi_\pi(s, u) \Big[\Lambda_\pi(s, u) \Big]^{d-1} ds du$$

cannot grow faster than $O(t)$, for large t. In a couple of cases in this example, i.e., $\pi_1 = \{\{1, 2, 3, 4\}\}$ and $\pi_2 = \{\{1, 2\}, \{3, 4\}\}$, we can show that $\Xi_\pi(s, u) \equiv 0$. Thus, these two partitions yield $A_\pi(t) \equiv 0$ and do not contribute to $\mathbb{E}_\omega\|\mathbf{X}_N(t)\delta_\mathbf{0}\|^2$. For each remaining partition $\pi \in \mathcal{P} \setminus \{\pi_1, \pi_2\}$, we want to estimate $A_\pi(t)$. To that end, recall the definition of

Table 1 Example Case $N = 1$

Partitions	Expectations	Upper bound of $c_\pi(t)$
{1, 2, 3, 4}	$\check{\mu}_\pi(s, u) = 1$	t^2
{1}{2, 3, 4}	$\check{\mu}_\pi(s, u) = \check{\mu}(s)\check{\mu}(-s) = \|\check{\mu}(s)\|^2$	$t \cdot \|\check{\mu}\|_2^2$
{1, 3, 4}{2}	$\check{\mu}_\pi(s, u) = \check{\mu}(-s)\check{\mu}(s) = \|\check{\mu}(s)\|^2$	$t \cdot \|\check{\mu}\|_2^2$
{1, 2, 4}{3}	$\check{\mu}_\pi(s, u) = \check{\mu}(u)\check{\mu}(-u) = \|\check{\mu}(u)\|^2$	$t \cdot \|\check{\mu}\|_2^2$
{1, 2, 3}{4}	$\check{\mu}_\pi(s, u) = \check{\mu}(-u)\check{\mu}(u) = \|\check{\mu}(u)\|^2$	$t \cdot \|\check{\mu}\|_2^2$
{1, 2}{3, 4}	$\check{\mu}_\pi(s, u) = 1$	t^2
{1, 3}{2, 4}	$\check{\mu}_\pi(s, u) = \check{\mu}(s-u)\check{\mu}(u-s) = \|\check{\mu}(s-u)\|^2$	$t \cdot \|\check{\mu}\|_2^2$
{1, 4}{2, 3}	$\check{\mu}_\pi(s, u) = \check{\mu}(u-s)\check{\mu}(s-u) = \|\check{\mu}(s-u)\|^2$	$t \cdot \|\check{\mu}\|_2^2$
{1}{2}{3, 4}	$\check{\mu}_\pi(s, u) = \check{\mu}(s)\check{\mu}(-s) = \|\check{\mu}(s)\|^2$	$t \cdot \|\check{\mu}\|_2^2$
{1}{2, 4}{3}	$\check{\mu}_\pi(s, u) = \check{\mu}(s)\check{\mu}(u)\check{\mu}(-s-u)$	$t \cdot \|\check{\mu}\|_2^2$
{1}{2, 3}{4}	$\check{\mu}_\pi(s, u) = \check{\mu}(s)\check{\mu}(-u)\check{\mu}(u-s)$	$t \cdot \|\check{\mu}\|_2^2$
{1, 4}{2}{3}	$\check{\mu}_\pi(s, u) = \check{\mu}(-s)\check{\mu}(u)\check{\mu}(s-u)$	$t \cdot \|\check{\mu}\|_2^2$
{1, 3}{2}{4}	$\check{\mu}_\pi(s, u) = \check{\mu}(-s)\check{\mu}(-u)\check{\mu}(s+u)$	$t \cdot \|\check{\mu}\|_2^2$
{1, 2}{3}{4}	$\check{\mu}_\pi(s, u) = \check{\mu}(u)\check{\mu}(-u) = \|\check{\mu}(u)\|^2$	$t \cdot \|\check{\mu}\|_2^2$
{1}{2}{3}{4}	$\check{\mu}_\pi(s, u) = \check{\mu}(s)\check{\mu}(-s)\check{\mu}(u)\check{\mu}(-u) = \|\check{\mu}(s)\|^2\|\check{\mu}(u)\|^2$	$\|\check{\mu}\|_2^2\|\check{\mu}\|_2^2$

$\check{\mathcal{U}}_\pi(s,u)$ in (3.35). If $\check{\mu} \in L^2(\mathbb{R})$, then there exist constants $\alpha > 0$ and $\beta > 0$ such that

$$|A_\pi(t)| \le \int_0^t \int_0^t \left|\check{\mathcal{U}}_\pi(s,u)\right| ds du \le \alpha \int_0^t \int_0^t |\check{\mu}_\pi(s,u)| ds du \le \beta c_\pi(t) = O(t),$$

for each $\pi \in \mathcal{P} \setminus \{\pi_1, \pi_2\}$. Thus, for each $\pi \in \mathcal{P}$ when $N = 1$, $|A_\pi(t)| \le \gamma_\pi t$ for some $\gamma_\pi \ge 0$, provided $\check{\mu} \in L^2(\mathbb{R})$.

Example 3.4 Equation (3.52) suggests that we should first study the one-dimensional problem, in which case $\xi(\cdot) = -2\cos(\cdot)$ by (3.6). Let $N = 2$ and suppose $\check{\mu} \in L^2(\mathbb{R})$. From Example 3.3, if π is such that $\check{\mu}_\pi(s,u) \not\equiv 1$, then

$$\int_0^t \int_0^t |\check{\mu}_\pi(s,u)| ds du \le ct,$$

for some constant $c > 0$. Hence, we need only consider partitions for which $\check{\mu}_\pi(s,u) \equiv 1$ and $\Xi_\pi(s,u) \not\equiv 0$. For $N = 2$, there are only five candidates for such partitions (see also Lemma 9); namely, $\pi_1 = \{\{1,3\},\{2,4\},\{5,7\},\{6,8\}\}$, $\pi_2 = \{\{1,3,5,7\},\{2,4,6,8\}\}$, $\pi_3 = \{\{1,3,5,7\},\{2,4\},\{6,8\}\}$, $\pi_4 = \{\{1,3,6,8\},\{2,4\},\{5,7\}\}$ and $\pi_5 = \{\{1,3,6,8\}, \{2,4,5,7\}\}$. For each of these partitions, we consider $\Xi_\pi(s,u)$ defined in (3.50). Define

$$I(t) := \int_{\mathbb{T}} e^{2ti\cos(x)} \cos(x)\, dx.$$

Note that $I(t) = iJ_1(2t)$, where $J_1(\cdot)$ is the Bessel function of the first kind of order one. Obviously, $I(-t) = \bar{I}(t)$. For each $y \in \mathbb{T}$, we observe that

$$\int_{\mathbb{T}} e^{2ti\cos(x-y)} \sin(x)\, dx = \int_{\mathbb{T}} e^{2ti\cos(x)} \sin(x+y)\, dx = \sin(y) I(t).$$

Then, for each of the five considered partitions above, we find that

$$\begin{aligned} \Xi_{\pi_1}(s,u) &= \Xi_{\pi_4}(s,u) = \Xi_{\pi_5}(s,u) = \frac{1}{2} |I(s/2)|^2 |I(u/2)|^2, \\ \Xi_{\pi_2}(s,u) &= \Xi_{\pi_3}(s,u) = 0. \end{aligned}$$

By the properties of Bessel functions (or equivalently by the stationary phase method [10]),

$$I(s) \approx i(\pi s)^{-1/2} \cos(2s - 3\pi/4),$$

for $s \gg 1$. Therefore, for each $\pi \in \mathcal{P}$ such that $\check{\mu}_\pi(s,u) \equiv 1$ and $\Xi_\pi(s,u) \not\equiv 0$, we obtain

$$|\Gamma_\pi(t)| \le \int_0^t \int_0^t |\Xi_\pi(s,u)| ds du \le Ct + \int_c^t \int_c^t |\Xi_\pi(s,u)| ds du \le Ct + c[\ln(t)]^2 = O(t),$$

for large t, by Remark 7. Therefore, we can conclude that, for each $\pi \in \mathcal{P}$ (for $N = 2$), $|A_\pi(t)| \le \gamma_\pi t$ for some $\gamma_\pi \ge 0$, provided $\check{\mu} \in L^2(\mathbb{R})$.

Remark 11 Examples 3.3 and 3.4 show that $\mathbb{E}_\omega \|\mathbf{X}_N(t)\delta_{\mathbf{0}}\|^2 \le ct$ when $N = 1, 2$ in any dimensions provided that $\check{\mu} \in L^2(\mathbb{R})$. It is worth noticing in these two examples that we obtain a rather good decay from $\Xi_\pi(s,u)$ whenever $\check{\mu}_\pi(s,u) \equiv 1$.

4 Oscillatory Integrals and Properties of the Phases

Our approach to determine the leading-term behavior of $r^2_{N,\delta_{\mathbf{0}}}(t) := \mathbb{E}_\omega \|\mathbf{X}_N(t)\delta_{\mathbf{0}}\|^2$ leads us to study an oscillatory integral whose overall phase is given by a linear combination of two functions. While both (3.46) and (3.52) are useful to prove Proposition 1, it suffices to only use (3.52). In terms of doing estimates, (3.46) allows us to first integrate with respect to s and u, which suggests the importance of having a nonvanishing phase. On the other hand, (3.52) permits us to study and estimate the oscillatory integrals $\Xi_\pi(s,u)$ in (3.50) and $\Lambda_\pi(s,u)$ in (3.51) corresponding to the problem in one dimension for fixed s and u, and then integrate with respect to s and u. In Appendix B, we state a result known as an extension of van de Corput's lemma [37] and modify it to our application. Then, we summarize some estimates on oscillatory integrals with two generic phases. In this section, we investigate the properties of the phases $f_1(N, \boldsymbol{\zeta}_1)$ and $g_1(N, \boldsymbol{\zeta}_1)$ after $\tilde{K}_\pi(\boldsymbol{\zeta}_{\sharp,1})$ is integrated out and apply the results in Appendix B to show that $|\Gamma_\pi(t)| = O_N(t)$ for some collection of partitions.

To apply the statements in Corollary 7 in Appendix B to estimate $\Gamma_\pi(t)$ or $A_\pi(t)$, one needs to show that the bound in (B.10) is satisfied when $\mathbb{I}(wt, vt) := \Xi_\pi(wt, vt)\Lambda_\pi(wt, vt)^{d-1}$. We will show that for a collection of partitions π, $|\Gamma_\pi(t)| = O_N(t)$ in dimensions $d \geq 3$ without any decay from the potential term $\check{\mathcal{U}}_\pi(s,u)$. In the following, we will classify these partitions and determine properties of the corresponding phases after $K_\pi(\boldsymbol{\zeta}_\sharp)$ has been integrated out.

4.1 Integrating over $K_\pi(\boldsymbol{\zeta}_\sharp)$

Definition 10 For a partition $\pi = \{S_1(\pi), \ldots, S_m(\pi)\} \in \mathcal{P}$, we say that $j \in \{1, \ldots, 4N\}$ is a *maximal element* of π if j is the greatest integer in some block $S_k(\pi)$ of π. An integer $j \in \{1, \ldots, 4N\}$ which is not a maximal element will be called a *non-maximal element*.

Definition 11 A variable, say ζ_j, corresponding to a maximal element j will be called a *maximal variable* associated with π. Otherwise, ζ_j will be called a *non-maximal variable*.

Remark 12 Accordingly, we can also define a *minimal element* of π and a *minimal variable* associated with π. By definition, every element in a singleton set is simultaneously a maximal element and a minimal element.

To estimate the oscillatory integral in (3.46) (resp. in (3.50) or (3.51)), we first integrate over the product of delta functions $K_\pi(\boldsymbol{\zeta}_\sharp)$ (resp. $\tilde{K}_\pi(\boldsymbol{\zeta}_{\sharp,1})$ in (3.50) or (3.51)). One approach to do this is as follows. First, we integrate over the delta functions corresponding to blocks with one element; that is, if π contains of a block, say, $S(\pi) = \{j\}$ for some $j \in \{1, \ldots, 4N\}$, then, in (3.46), we integrate over the delta function $\delta(\zeta_j)$. Next, for blocks $S_l(\pi)$ such that $|S_l(\pi)| > 1$, we replace all of the maximal variables by the non-maximal ones. Alternatively, we can also replace the minimal variables by the non-minimal ones. In the end, the remaining integral is over $\mathbb{T}^{d(4N+1-|\pi|)}$ in (3.46) (resp. $\mathbb{T}^{4N+1-|\pi|}$ in (3.50) or (3.51)). Unless otherwise stated, our convention is to replace each maximal variable of π by a linear combination of the non-maximal variables corresponding to the same block. For instance, if π contains a block $S(\pi) = \{1, N, 2N, 3N+2\}$, then ζ_{3N+2} is replaced by $-\zeta_1 - \zeta_N - \zeta_{2N}$. It should be noted that ζ_0 never appears in the delta functions, see (3.41); thus, by default, it is always a non-maximal variable. On the other hand, ζ_{4N} is always a maximal variable and gets integrated out. After this process, the non-maximal variables ζ_j whose blocks contain more than one element remain independent, and are used as variables on which the phases depend.

Let $\pi \in \mathcal{P}$ be fixed. In (3.46), the two phases $\tilde{\mathcal{F}}(N, \boldsymbol{\zeta})$ and $\tilde{\mathcal{G}}(N, \boldsymbol{\zeta})$ are given by

$$\tilde{\mathcal{F}}(N, \boldsymbol{\zeta}) = \frac{1}{N}\left(\sum_{j=1}^{N-1} \xi(\zeta_0 + \zeta_1 + \ldots + \zeta_j) - \sum_{j=N+1}^{2N-1} \xi(\zeta_0 + \zeta_1 + \ldots + \zeta_j)\right),$$

$$\tilde{\mathcal{G}}(N, \boldsymbol{\zeta}) = \frac{1}{N}\left(\sum_{j=2N+1}^{3N-1} \xi(\zeta_0 + \zeta_1 + \ldots + \zeta_j) - \sum_{j=3N+1}^{4N-1} \xi(\zeta_0 + \zeta_1 + \ldots + \zeta_j)\right),$$

subjected to the constraint equations given by the delta functions in (3.41), where $\zeta_j \in \mathbb{T}^d$, for $j = 0, \ldots, 4N$. As already noted, $\tilde{\mathcal{F}}(N, \boldsymbol{\zeta})$ and $\tilde{\mathcal{G}}(N, \boldsymbol{\zeta})$ can be further decomposed into

$$\tilde{\mathcal{F}}(N, \boldsymbol{\zeta}) = \sum_{l=1}^{d} f_l(N, \boldsymbol{\zeta}_l), \tag{4.1}$$

$$\tilde{\mathcal{G}}(N, \boldsymbol{\zeta}) = \sum_{l=1}^{d} g_l(N, \boldsymbol{\zeta}_l), \tag{4.2}$$

where $f_l(N, \boldsymbol{\zeta}_l)$ and $g_l(N, \boldsymbol{\zeta}_l)$, given respectively by (3.47) and (3.48), only depend on $\boldsymbol{\zeta}_l = (\zeta_{0,l}, \zeta_{1,l}, \ldots, \zeta_{4N,l})$ and will be subjected to the constraint equations given by the delta functions in (3.42). Thus, $f_j(N, \boldsymbol{\zeta}_j)$ and $f_l(N, \boldsymbol{\zeta}_l)$, when $j \neq l$, are functions of different independent variables. Respectively, the analogous statements hold for $g_l(N, \boldsymbol{\zeta}_l)$. Moreover, the factor $-2/N$ only affects the properties of $f_l(N, \boldsymbol{\zeta}_l)$ and $g_l(N, \boldsymbol{\zeta}_l)$ in a trivial manner. Consequently, without loss of generality, we only need to study the properties of $f_1(N, \boldsymbol{\zeta}_1)$ and $g_1(N, \boldsymbol{\zeta}_1)$ without the factor $-2/N$, which will be denoted by $f(\zeta)$ and $g(\zeta)$, respectively. Our main objective for the rest of this section is to study the properties of $f(\zeta)$ and $g(\zeta)$.

4.2 One Dimensional Problem

For a given $\pi \in \mathcal{P}$, after the delta functions have been integrated out, the phases $f(\zeta) = f_\pi(\zeta)$ and $g(\zeta) = g_\pi(\zeta)$ can be expressed explicitly as

$$f_\pi(\zeta) = \sum_{i=1}^{N-1} \cos(\kappa_i) - \sum_{i=N+1}^{2N-1} \cos(\kappa_i), \tag{4.3}$$

$$g_\pi(\zeta) = \sum_{i=2N+1}^{3N-1} \cos(\kappa_i) - \sum_{i=3N+1}^{4N-1} \cos(\kappa_i), \tag{4.4}$$

where

$$\kappa_i = -\sum_{j=0}^{i} \sigma_{i,j}\zeta_j, \tag{4.5}$$

with $\sigma_{i,j} = 0$ or 1. It should be emphasized that hereafter $\zeta_i, \kappa_i \in \mathbb{T}$ are all one dimensional, and $\zeta := (\zeta_0, \zeta_1, \dots, \zeta_{4N}) := (\zeta_0, \zeta_\sharp) \in \mathbb{T} \times \mathbb{T}^{4N}$. We note that κ_i in (4.5) now only depends on the non-maximal variables ζ_j, hence so do $f_\pi(\zeta)$ and $g_\pi(\zeta)$, and we can consider them as functions on $\mathbb{T}^{4N+1-|\pi|}$. Henceforth, we refer to κ_i as an *argument* of $\cos(\kappa_i)$ rather than an independent variable. To be precise, let $z \in \mathbb{T}^{4N+1-|\pi|}$ denote a $(4N+1-|\pi|)$-tuple of non-maximal variables ζ_j; that is,

$$z := (z_0, z_{i_1}, z_{i_2}, \dots, z_{i_{4N-|\pi|}}),$$

where $1 \le i_1 < i_2 < \dots < i_{4N-|\pi|} < 4N$ are all non-maximal elements of π, $z_0 = \zeta_0$, and $z_j = \zeta_j$ if j is a non-maximal element. As such, we can write $f_\pi(\zeta) = \tilde{f}_\pi(z)$ and $g_\pi(\zeta) = \tilde{f}_\pi(z)$. See Example 4.3. With the integrand $\Psi(\kappa_N, \kappa_{3N}) = \Psi_\pi(\kappa_N, \kappa_{3N}) = \sin(\kappa_N)\sin(\kappa_{3N})$, we can rewrite $\Xi_\pi(s,u)$ and $\Lambda_\pi(s,u)$ as

$$\Xi_\pi(s,u) := \int_{\mathbb{T}^{4N+1-|\pi|}} e^{is\tilde{f}_\pi(z)} e^{iu\tilde{g}_\pi(z)} \Psi_\pi(\kappa_N, \kappa_{3N})\, dz, \tag{4.6}$$

$$\Lambda_\pi(s,u) := \int_{\mathbb{T}^{4N+1-|\pi|}} e^{is\tilde{f}_\pi(z)} e^{iu\tilde{g}_\pi(z)}\, dz. \tag{4.7}$$

Definition 12 After integrating over the delta functions, some κ_i's defined in (4.5) may be equal. By convention, if $\kappa_i = \kappa_j$ and $i < j$, then κ_j is replaced by κ_i.

(i) We say that κ_j *survives* if it is not equal to any κ_i for $i < j$.
(ii) We say that ζ_j *appears* in κ_i, for $i, j \in \{1, \dots, 4N\}$, if κ_i survives and is a function of ζ_j, i.e., $\sigma_{i,j} = 1$ in (4.5).
(iii) We say that $\cos(\kappa_j)$ is *present* in $f_\pi(\zeta)$ if there does not exist $\cos(\kappa_i)$, with $i, j \in \{1, \dots, 2N-1\} \setminus \{N\}$, such that $\cos(\kappa_i) - \cos(\kappa_j) \equiv 0$. Analogously, we say that $\cos(\kappa_j)$ is *present* in $g_\pi(\zeta)$ if there does not exist $\cos(\kappa_i)$, with $i, j \in \{2N+1, \dots, 4N-1\} \setminus \{3N\}$, such that $\cos(\kappa_i) - \cos(\kappa_j) \equiv 0$.

Remark 13 The definition guarantees that the *surviving* κ_i's in (4.5) are all distinct, and indeed, independent because each of them is a linear combination of non-maximal variables associated with π, and no pair of such κ_i is expressed in terms of the same set of the non-maximal variables ζ_j. It is clear that $f_\pi(\zeta)$ depends on ζ_j if and only if $\tilde{f}_\pi(z)$ depends on z_j, and the same relationship holds for $g_\pi(\zeta)$ and $\tilde{g}_\pi(z)$. Although we could use the surviving κ_j as our independent variables, we will work with the non-maximal variables ζ_j or, equivalently, z_j.

4.3 Some Combinatoric Structures

In (4.6) and (4.7), the properties of the phases $\tilde{f}_\pi(z)$ and $\tilde{g}_\pi(z)$ depend critically on the structure of π. One useful technique to learn about these structures is to represent and classify partitions by their graphs [8, 12]. Even though our classification is not in the same spirit as in [8, 12], we find it useful to represent our partitions in similar fashion. Unlike the method in [8, 12], our method contains two time parameters: s and u, and we must keep track of $\pm s$ and $\pm u$.

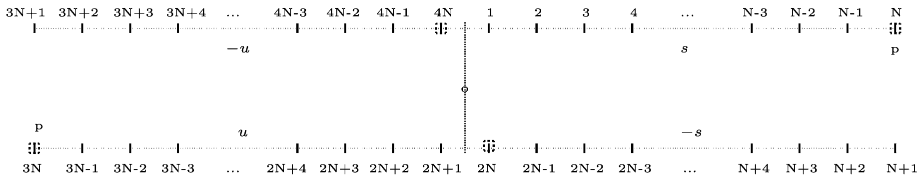

Fig. 1 A base diagram

Example 4.1 $\pi = \{\{1, N\}, \{2, 2N, 3N\}, \{3\}, \dots, \{N-1\}, \{N+1\}, \dots, \{2N-1\}, \{2N+1\}, \dots, \{3N-1\}, \{3N+1\}, \dots, \{4N\}\}$.

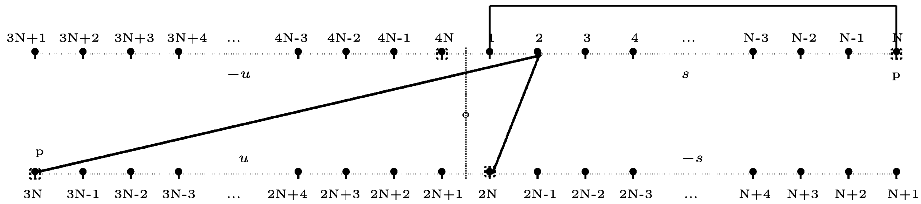

Fig. 2 An example of a representation of a partition by its graph

In the base diagram in Fig. 1, there are four horizontal dashed lines, called *propagation lines*, corresponding to the parameters $\pm s, \pm u$. The lone vertical line, dividing the diagram into the right-hand side and the left-hand side, is used to indicate that $\cos(\kappa_i)$, for $i \in \{1, \dots, 2N-1\} \setminus \{N\}$, appears in $\tilde{f}_\pi(z)$, and that $\cos(\kappa_i)$, for $i \in \{2N+1, \dots, 4N-1\} \setminus \{3N\}$, appears in $\tilde{g}_\pi(z)$. The numbers on the dashed lines represent indices of the lattice points at which the potential terms are being evaluated. If two indices i and j belong to the same block of π, then they are connected in the diagram by a solid line. Each block with a single element is presented by a solid node; see Fig. 2. The letter p at the N-th and $3N$-th nodes is a reminder that the function $\Psi_\pi(\cdot,\cdot)$ involves κ_N and κ_{3N}, i.e., that the N- and $3N$-th terms in the convolution are the Fourier transform of quantities involving the momentum operator $\mathbf{p}$, see (3.20) in Lemma 3. In addition, the boxes around the N-th, $2N$-th, $3N$-th and $4N$-th nodes signify that $\cos(\kappa_N), \cos(\kappa_{2N}), \cos(\kappa_{3N})$ and $\cos(\kappa_{4N})$ do not appear in $\tilde{f}_\pi(z)$ and $\tilde{g}_\pi(z)$.

While it is not absolutely essential in the proof of our result, we find it convenient to view each partition from its graph. In addition to keeping track of variables such as t_j defined in (3.21) and helping to integrate over $\tilde{K}_\pi(\zeta_\sharp)$, the diagram also helps to construct examples.

Remark 14 Examples 3.3 and 3.4 suggest that there are many partitions π (such as those with $\Xi_\pi(s,u) \equiv 0$) which do not contribute to $\mathbb{E}_\omega \|\mathbf{X}_N(t)\delta_\mathbf{0}\|^2$. It is interesting and useful to identify all such partitions. In general, we hope to divide partitions into smaller subcollections on which we can estimate $A_\pi(t)$ and $\Gamma_\pi(t)$ based on the common structures of each subcollection—an approach which was introduced by [8, 12].

Definition 13 (Some Classifications of Graphs) Let $\pi = \{S_1(\pi), S_2(\pi), \dots, S_m(\pi)\} \in \mathcal{P}$.

(i) A partition π has a *separated* graph if there does not exist a pair of elements i, j with $1 \le i \le 2N$ and $2N+1 \le j \le 4N$ such that i and j belong to the same block.
(ii) A partition π has a *pairing* graph if $|S_k(\pi)| = 2$ for each $k = 1, \dots, m$.

(iii) A partition π has a *ladder graph on the right-hand side* if it contains blocks of the form $\{1, 2N\}, \{2, 2N-1\}, \ldots, \{k, 2N+1-k\}, \ldots, \{N, N+1\}$ while the blocks containing elements from {2N+1,...,4N} are arbitrary. Respectively, π has a *ladder graph on the left-hand side* if it contains blocks of the form $\{2N+1, 4N\}, \{2N+2, 4N-1\}, \ldots, \{2N+k, 4N+1-k\}, \ldots, \{3N, 3N+1\}$ while the blocks containing elements from $\{1, \ldots, 2N\}$ are arbitrary. π has a *ladder graph* if it simultaneously has a ladder graph on the left- and right-hand sides.

These are examples of classifications of graphs, which help to study the properties of $\tilde{f}_\pi(z)$ and $\tilde{g}_\pi(z)$, hence to estimate $A_\pi(t)$ and $\Gamma_\pi(t)$.

Example 4.2 A ladder graph on one side.

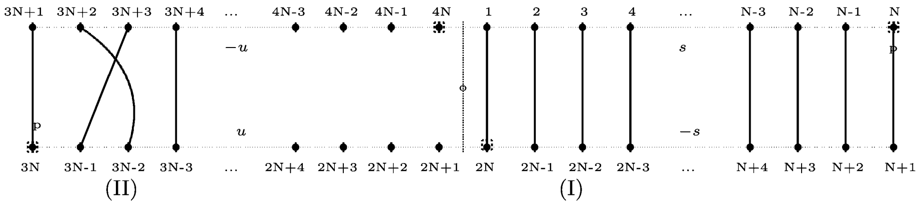

Fig. 3 A ladder graph on the right-hand side

Remark 15

(i) It is clear that a ladder graph is a special case of a pairing graph.
(ii) A ladder graph on the right-hand side or on the left-hand side is a special case of a separated graph.
(iii) If π has a ladder graph, then $\Xi_\pi(s, u) \equiv 0$, so $\Gamma_\pi(t) = A_\pi(t) = 0$. This is a special case of a more general result proved below.

Lemma 9 *If either* N *and* $N+1$ *or* $3N$ *and* $3N+1$ *belong to the same block of* π, *then* $\Xi_\pi(s, u) \equiv 0$.

Proof Let $\pi \in \mathcal{P}$ be fixed. Suppose $3N$ and $3N+1$ belong to the same block of π. If $3N+1$ is a maximal element, then $3N$ is a non-maximal element, so ζ_{3N} is a non-maximal variable of which both $f_\pi(\zeta)$ and $g_\pi(\zeta)$ are independent. Therefore, $\tilde{f}_\pi(z)$ and $\tilde{g}_\pi(z)$ are independent of z_{3N}. However, $\Psi_\pi(\kappa_N, \kappa_{3N})$ depends on z_{3N} because

$$\Psi_\pi(\kappa_N, \kappa_{3N}) = \sin(z_0 + \ldots + z_\nu)\sin(z_0 + \ldots + z_{3N}),$$

for some non-maximal element ν with $1 \le \nu \le N$; cf. (4.5). As a result, integrating (4.6) with respect to z_{3N} yields $\Xi_\pi(s, u) \equiv 0$.

If $3N+1$ is a non-maximal element, then there exists a maximal element $j > 3N+1$ such that ζ_{3N} and ζ_{3N+1} do not appear in κ_i for $i \ge j$. (Recall that, before integrating over the delta functions, $\zeta_{3N} + \zeta_{3N+1}$ appears in each κ_i for $i \ge 3N+1$). Let $w_{3N} := z_{3N} = \zeta_{3N}$ and $w_{3N+1} := z_{3N} + z_{3N+1} = \zeta_{3N} + \zeta_{3N+1}$. Then, by (4.4), $\tilde{g}_\pi(z) := g_\pi(\zeta)$ depends on w_{3N+1}, but does not depend on w_{3N}. However, $\Psi_\pi(\kappa_N, \kappa_{3N})$ does depend on w_{3N} as

$$\Psi_\pi(\kappa_N, \kappa_{3N}) = \sin(z_0 + \ldots + z_\nu)\sin(z_0 + \ldots + w_{3N}),$$

for some non-maximal element ν with $1 \leq \nu \leq N$. It is clear that $\tilde{f}_\pi(z)$ is independent of z_{3N}, hence independent of w_{3N}. As a result, the change of variables allows us to integrate (4.6) with respect to w_{3N} to obtain $\Xi_\pi(s,u) \equiv 0$.

Similarly, if N and $N+1$ belong to the same block of π and $N+1$ is a maximal element, then $\tilde{f}_\pi(z)$ and $\tilde{g}_\pi(z)$ are independent of $z_N = \zeta_N$. Moreover,

$$\Psi_\pi(\kappa_N, \kappa_{3N}) = \sin(z_0 + \ldots + z_N)\sin(z_0 + \ldots + z_\nu),$$

for some non-maximal variable ν with $1 \leq \nu \leq 3N$, where z_N does not appear in the argument of the second *sine* term. Thus, the same conclusion follows by integrating (4.6) with respect to z_N. If N and $N+1$ belong to the same block and $N+1$ is a non-maximal element, then there exists a maximal element $j > N+1$ such that ζ_N and ζ_{N+1} do not appear in κ_i for $i \geq j$. Let $w_N := z_N = \zeta_N$ and $w_{N+1} := z_N + z_{N+1} = \zeta_N + \zeta_{N+1}$. Then, as before, both $\tilde{f}_\pi(z)$ and $\tilde{g}_\pi(z)$ are independent of w_N, and only one factor of $sine$ in $\Psi_\pi(\kappa_N, \kappa_{3N})$ does depend on w_N. Hence, the same conclusion follows by integrating (4.6) with respect to w_N. This completes the proof of the lemma. □

Remark 16 It follows from the delta functions in (3.42) that, for each partition $\pi \in \mathcal{P}$, we always have $\zeta_1 + \zeta_2 + \ldots + \zeta_{4N} = 0$, which implies $\kappa_{4N} = \kappa_0 = -\zeta_0$. If, in addition, π has a separated graph, then we also have that $\zeta_1 + \zeta_2 + \ldots + \zeta_{2N} = 0$, and $\zeta_{2N+1} + \zeta_{2N+2} + \ldots + \zeta_{4N} = 0$, which means that $\kappa_{2N} = -\zeta_0 = \kappa_0 = \kappa_{4N}$. The converse is also true; that is, if $\kappa_{4N} = \kappa_{2N} = \kappa_0 = -\zeta_0$, then $\zeta_1 + \zeta_2 + \ldots + \zeta_{2N} = 0$ and $\zeta_{2N+1} + \zeta_{2N+2} + \ldots + \zeta_{4N} = 0$, which implies that π has a separated graph.

If π has a separated graph, then $\tilde{f}_\pi(z)$ and $\tilde{g}_\pi(z)$ are functions of different independent variables, except for z_0. Therefore, the corresponding oscillatory integrals in (4.6), and (4.7) simplify to the product of two oscillatory integrals, each with a single phase, to which one can apply the standard stationary phase and non-stationary principles to estimate $A_\pi(t)$ and $\Gamma_\pi(t)$.

Lemma 10 *If π has a ladder graph on the right-hand side, then $\tilde{f}_\pi(z) \equiv 0$. If π has a ladder graph on the left-hand side, then $\tilde{g}_\pi(z) \equiv 0$. Consequently, if π has a ladder graph, then $\tilde{f}_\pi(z) \equiv \tilde{g}_\pi(z) \equiv 0$.*

Proof It is enough to prove the corresponding statements for $f_\pi(\zeta)$ and $g_\pi(\zeta)$. In order to have a ladder graph on either side or on both sides, π must have a separated graph, which implies that $f_\pi(\zeta)$ depends on the non-maximal variables ζ_j with $1 \leq j \leq 2N$, while $g_\pi(\zeta)$ depends on the non-maximal variables ζ_j with $2N+1 \leq j \leq 4N$. Suppose π has a ladder graph on the right-hand side, then $\zeta_0, \zeta_1, \ldots, \zeta_N$ are non-maximal variables and $\zeta_{N+1}, \ldots, \zeta_{2N}$ are maximal variables. After integrating over the delta functions, the arguments κ_i in (4.5) are given exactly by

$$\kappa_i = -\sum_{j=0}^{i} \zeta_j,$$

for $i = 0, \ldots, N$. Since π has a ladder graph on the right-hand side, the arguments of the delta functions in (3.42) imply $\zeta_j + \zeta_{2N-j+1} = 0$ for each $j = 1, \ldots, N$; in particular, $\zeta_N + \zeta_{N+1} = 0$, which yields

$$\kappa_{N+1} = \kappa_{N-1} - \zeta_N - \zeta_{N+1} = \kappa_{N-1}.$$

Moreover, $\zeta_{N-1}+\zeta_{N+2}=0$, which gives

$$\kappa_{N+2}=\kappa_{N+1}-\zeta_{N+2}=\kappa_{N-2}-\zeta_{N-1}-\zeta_{N-2}=\kappa_{N-2}.$$

By induction, $\kappa_{N+j}=\kappa_{N-j}$ for each $j=1,\dots,N$. Therefore, the *cosine* terms in the expression (4.3) cancel out in pairs, which results in $f_\pi(\zeta)\equiv 0$. Therefore, $\tilde{f}_\pi(z)\equiv 0$.

Now suppose that π has ladder graph on the left-hand side, then

$$\kappa_i=-\zeta_0-\sum_{j=2N+1}^{i}\zeta_j$$

for each i with $2N+1\le i\le 3N$. Since $\zeta_{3N+j}+\zeta_{3N-j+1}=0$, the above argument applies, and we can conclude that $\kappa_{3N+j}=\kappa_{3N-j}$. Therefore, $\tilde{g}_\pi(z)=g_\pi(\zeta)=0$, which proves the lemma. □

The converse is not true as is evident from $\pi=\{\{1,5,8\},\{2\},\{3\},\{4\},\{6\},\{7\}\}$ when $N=2$, which illustrates that $\tilde{f}_\pi(z)$ and $\tilde{g}_\pi(z)$ can vanish identically even though π does not have a ladder on either side.

Definition 14 We say that a partition $\pi=\{S_1(\pi),S_2(\pi),\dots,S_m(\pi)\}$ is *even* if $|S_l|\in 2\mathbb{N}$ for all $1\le l\le m$.

Definition 15 Given the collection of partitions $\mathcal{P}$, let $\mathcal{P}_E$ denote the subcollection of even partitions, and $\mathcal{P}_0$ the subcollection of partitions $\pi=\{S_1(\pi),S_2(\pi),\dots,S_m(\pi)\}$ such that $\sum_{j\in S_k(\pi)}t_j\equiv 0$ for each $k=1,\dots,m$, where t_j is defined in (3.21).

It is clear that $\mathcal{P}_0\subset\mathcal{P}_E$, and that if $\pi\in\mathcal{P}_0$, then $\check{\mu}_\pi(s,u)\equiv 1$. The above classification suggests that when $\pi\notin\mathcal{P}_0$, then we immediately obtain some decay from $\check{\mathcal{U}}_\pi(s,u)$ provided that $\check{\mu}\in L^2(\mathbb{R})$, as seen in Examples 3.3 and 3.4. A rigorous proof of this statement will be given by Lemma 16 in Sect. 5. Showing that $|A_\pi(t)|=O_N(t)$ or $|\Gamma_\pi(t)|=O_N(t)$ for $\pi\in\mathcal{P}_0$ is more technical.

Lemma 11 *$\pi\notin\mathcal{P}_0$ does not have a subpartition $\pi'\in\mathcal{P}_0$.*

Proof Since $\pi\notin\mathcal{P}_0$, then there exists a block, say $S_k(\pi)$, such that $\sum_{l\in S_k(\pi)}t_l\neq 0$. Consider a partition of this block; suppose it consists of sets $S_{k_1},S_{k_2},\dots,S_{k_n}$, where $\bigcup_{j=1}^n S_{k_j}=S_k(\pi)$ and $S_{k_j}\cap S_{k_l}=\emptyset$ for $j\neq l$. If $\sum_{l\in S_{k_j}}t_l=0$, for all $j=1,\dots,n$, then $\sum_{l\in S_k(\pi)}t_l=\sum_{j=1}^n\sum_{l\in S_{k_j}}t_l=0$, which leads to a contradiction. Therefore, there exists a block S_{k_j} such that $\sum_{l\in S_{k_j}}t_l\neq 0$, which implies that if $\pi\notin\mathcal{P}_0$, then its subpartition of π does not belong to $\mathcal{P}_0$. □

Corollary 2 *In (3.35), there exists $\pi'\prec\pi$ such that $\check{\mu}_{\pi'}(s,u)\equiv 1$ if and only if $\pi\in\mathcal{P}_0$. Thus, if $\pi\notin\mathcal{P}_0$, then $\check{\mathcal{U}}_\pi(s,u)$ does not have a constant term independent of s and u.*

Proof If $\pi\in\mathcal{P}_0$, it is obvious that $\check{\mu}_\pi(s,u)\equiv 1$. If $\pi\notin\mathcal{P}_0$, then $\check{\mu}_{\pi'}(s,u)\equiv 1$ for some subpartition $\pi'\prec\pi$ only if $\pi'\in\mathcal{P}_0$. This is impossible by Lemma 11. Thus, if $\pi\notin\mathcal{P}_0$, then every term in the sum on the right-hand side of (3.35) for $\check{\mathcal{U}}_\pi(s,u)$ depends on s or u, which proves the claim. □

4.4 Some Properties of Phases

In this section, we will study some properties of $\tilde{f}_\pi(z)$ and $\tilde{g}_\pi(z)$. We have already learned from Lemma 10 that $\tilde{f}_\pi(z)$ and $\tilde{g}_\pi(z)$ can be identically zero for some collection of partitions.

Example 4.3 Let $\pi = \{\{1,4\},\{2,5\},\{3,6\},\{7,10\},\{8,11\},\{9,12\}\}$ when $N = 3$. Here,

$$\zeta := (\zeta_0, \zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6, \zeta_7, \zeta_8, \zeta_9, \zeta_{10}, \zeta_{11}, \zeta_{12}) \in \mathbb{T}^{13},$$

$$z := (\zeta_0, \zeta_1, \zeta_2, \zeta_3, \zeta_7, \zeta_8, \zeta_9) = (z_0, z_1, z_2, z_3, z_7, z_8, z_9) \in \mathbb{T}^7.$$

In terms of non-maximal variables, $\tilde{f}(z)$ and $\tilde{g}(z)$ can be written as

$$\tilde{f}_\pi(z) := \cos(z_0 + z_1) + \cos(z_0 + z_1 + z_2) - \cos(z_0 + z_2 + z_3) - \cos(z_0 + z_3),$$

$$\tilde{g}_\pi(z) := \cos(z_0 + z_7) + \cos(z_0 + z_7 + z_8) - \cos(z_0 + z_8 + z_9) - \cos(z_0 + z_9).$$

Define $F_\pi(z, s, u) := s\tilde{f}_\pi(z) + u\tilde{g}_\pi(z)$. We observe that if $\tilde{z} = (a, b, \pi, b, b, \pi, b) \in \mathbb{T}^7$ with $a + b = \pm\frac{\pi}{2}$, then $\tilde{f}_\pi(\tilde{z}) = \tilde{g}_\pi(\tilde{z}) = 0$ and $\nabla_z \tilde{f}_\pi(\tilde{z}) = \nabla_z \tilde{g}_\pi(\tilde{z}) = 0$. It is also easy to check that all second partial derivatives of $F_\pi(z, s, u)$ with respect to z vanish at $\tilde{z}$ for all $s, u \geq 0$. This example shows that there exists a partition such that $\tilde{f}_\pi(z)$ and $\tilde{g}_\pi(z)$ are not identically zero, yet they and their Hessian matrices can simultaneously vanish at a critical point.

Remark 17 There exists a partition π such that the set of critical points of $\tilde{f}_\pi(z)$ and $\tilde{g}_\pi(z)$ is not discrete. In fact, they are not necessarily manifolds. As noted, the set of critical points of $F_\pi(z, s, u) := s\tilde{f}_\pi(z) + u\tilde{g}_\pi(z)$ depends on the parameters s and u. In [39], it is shown that if $(s_0, u_0, \tilde{z}) \in [0, 1] \times [0, 1] \times \mathbb{T}^{4N+1-|\pi|}$ such that the Hessian matrix of $s_0\tilde{f}_\pi(\tilde{z}) + u_0\tilde{g}_\pi(\tilde{z})$ has rank at least one, then $|\Gamma_\pi(t)| \leq ct$, for some $c > 0$. However, it is clear from Lemma 10 and Example 4.3 that such a condition is not always satisfied. It is not clear how to estimate oscillatory integrals in (B.1) whose two phases along with their first and second derivatives vanish at the same point.

Definition 16 Let $\mathfrak{P}$ denote the collection of partitions $\pi = \{S_1(\pi), \ldots, S_m(\pi)\} \in \mathcal{P}$ with the following properties:

(i) N and $N + 1$ do not belong to the same block;
(ii) $3N$ and $3N + 1$ do not belong to the same block;
(iii) $|S_k(\pi)| > 1$ for each $k = 1, \ldots, m$;
(iv) $1, \ldots, N$ and $2N + 1, \ldots, 3N$ are non-maximal elements.

Remark 18 From Definition 16, we can deduce the following.

(i) $z = (z_0, z_1, \ldots, z_N, z_{k_1}, \ldots, z_{k_p}, z_{2N+1}, z_{2N+1}, \ldots, z_{3N}, z_{l_1}, \ldots, z_{l_q})$, where $N + 1 \leq k_1 < k_2 < \ldots < k_p \leq 2N$ and $3N + 1 \leq l_1 < l_2 < \ldots < l_q < 4N$ are non-maximal elements.
(ii) Consider $f_\pi(\zeta)$ in (4.3) and $g_\pi(\zeta)$ in (4.4). Since $|S_k(\pi)| > 1$ for each $k = 1, \ldots, m$, if j is a maximal element, i.e., ζ_j is a maximal variable, then there exists a non-maximal element $l \in \{1, \ldots, j\}$ such that the non-maximal variable ζ_l does not appear in κ_i for each $i \geq j$; hence the arguments κ_i do not depend on ζ_l. It is more evident to see this from a graph viewpoint, using the diagram in Fig. 3.

(iii) κ_{N+1} is distinct from any other κ_i, with $i \leq N$. If $N+1$ is non-maximal, then by (4.5),

$$\kappa_i = -\sum_{j=0}^{i} \zeta_j,$$

for each $i = 0, 1, \ldots, N+1$. In particular, ζ_{N+1} appears in κ_{N+1}, but it does not appear in κ_i, for any $i = 1, \ldots, N$. If $N+1$ is a maximal element, then there exists a non-maximal element $j \in \{1, \ldots, N-1\}$ such that ζ_j does not appear in κ_{N+1} because $|S_k(\pi)| > 1$ for each $k = 1, \ldots, |\pi|$ and N does not belong to the same block as $N+1$. This implies that $\kappa_{N+1} \neq \kappa_N$ because ζ_j does appear in κ_N. To see that $\kappa_{N+1} \neq \kappa_i$ for any $i < N$, we again consider κ_i in (4.5) and observe that ζ_N does not appear in κ_i for any $i < N$, but it does appear in κ_{N+1} as

$$\kappa_{N+1} = -\sum_{j=0}^{N+1} \sigma_{N+1,j}\zeta_j = -\sum_{j=0}^{N+1} \sigma_{N+1,j}\zeta_j - \zeta_N.$$

This proves that κ_{N+1}, which depends on ζ_N, is distinct from any other κ_i, for $i \leq N$. Consequently, $\cos(\kappa_{N+1})$ will be present in (4.3).

(iv) Similarly, κ_{3N+1}, which depends on ζ_{3N}, is distinct from other κ_i, where $2N+1 \leq i \leq 3N$, and $\cos(\kappa_{3N+1})$ will be present in (4.4).

Lemma 12 *Suppose $\pi \in \mathfrak{P}$. There exist multi-indices α and β with $|\alpha| = 2$, and $|\beta| = 3$ such that*

$$|\partial_z^\alpha \tilde{g}_\pi(z)|^2 + |\partial_z^\beta \tilde{g}_\pi(z)|^2 = 1, \tag{4.8}$$

$$\partial_z^\alpha \tilde{f}_\pi(z) = \partial_z^\beta \tilde{f}_\pi(z) = 0. \tag{4.9}$$

Proof Fix $\pi \in \mathfrak{P}$. Let p denote the greatest non-maximal element of π. By definition, $p+1$ is a maximal element, and $p \geq 3N$ because $\pi \in \mathfrak{P}$. Recall that $\tilde{f}_\pi(z) = f_\pi(\zeta)$ given by (4.3), and that $\tilde{g}_\pi(z) = g_\pi(\zeta)$ given by (4.4).

If $p = 3N$, then $3N+1, \ldots, 4N$ are all maximal elements. From Remark 18(iii), $\cos(\kappa_{3N+1})$ is present in (4.4), and κ_{3N+1} depends on ζ_{3N}, one of the non-maximal variables. Since $3N+2$ is also a maximal element and $|S_k(\pi)| > 1$, for each $k = 1, \ldots, |\pi|$, there exists a non-maximal element $l \in \{1, \ldots, 3N\}$ such that the non-maximal variable ζ_l does not appear in κ_i when $i \geq 3N+2$, by Remark 18(ii). However, ζ_l does appear in κ_{3N+1}. As a result,

$$\frac{\partial^2}{\partial z_{3N} \partial z_l} \tilde{g}_\pi(z) = \frac{\partial^2}{\partial \zeta_{3N} \partial \zeta_l} g_\pi(\zeta) = \cos(\kappa_{3N+1})\Big|_z$$

and

$$\frac{\partial^3}{\partial z_{3N}^2 \partial z_l} \tilde{g}_\pi(z) = \frac{\partial^3}{\partial \zeta_{3N}^2 \partial \zeta_l} g_\pi(\zeta) = \sin(\kappa_{3N+1})\Big|_z$$

satisfy (4.8).

Now suppose $p > 3N$. Since $p+1$ is a maximal element, there exists a non-maximal element $l \in \{1, \ldots, p\}$ such that the non-maximal variable ζ_l does not appear in κ_i for $i \geq p+1$ by Remark 18(ii). Moreover, since ζ_p is a non-maximal variable, $\cos(\kappa_p)$ is present in (4.4). Therefore,

$$\frac{\partial^2}{\partial z_p \partial z_l}\tilde{g}_\pi(z) = \frac{\partial^2}{\partial \zeta_p \partial \zeta_l} g_\pi(\zeta) = \cos(\kappa_p)\Big|_z$$

and

$$\frac{\partial^3}{\partial z_p^2 \partial z_l}\tilde{g}_\pi(z) = \frac{\partial^3}{\partial \zeta_p^2 \partial \zeta_l} g_\pi(\zeta) = \sin(\kappa_p)\Big|_z$$

satisfy (4.8).

In both cases, ζ_p is one of the non-maximal variables of which $f_\pi(\zeta)$ is independent; thus, it is immediate that

$$\frac{\partial}{\partial z_p}\tilde{f}_\pi(z) = \frac{\partial}{\partial \zeta_p} f_\pi(\zeta) = 0.$$

This completes the proof of the lemma. □

Corollary 3 *Let $\pi \in \mathfrak{P}$. Then, with $k = 3$ and $vt \geq 1$,*

$$|\Xi_\pi(wt, vt)| \leq C_1 t^{-1/k} v^{-1/k}, \tag{4.10}$$

$$|\Lambda_\pi(wt, vt)| \leq C_2 t^{-1/k} v^{-1/k}, \tag{4.11}$$

where C_1 and C_2 are constants independent of w, v, and t.

Proof With $s = wt$, $u = vt$, and using (4.6) and (4.7), we can write

$$\Xi_\pi(wt, vt) = \int_{\mathbb{T}^{4N+1-|\pi|}} e^{itF_\pi(z,w,v)} \Psi_\pi(\kappa_N, \kappa_{3N})\, dz,$$

$$\Lambda_\pi(wt, vt) = \int_{\mathbb{T}^{4N+1-|\pi|}} e^{itF_\pi(z,w,v)}\, dz,$$

with $F_\pi(z, w, v) = w\tilde{f}_\pi(z) + v\tilde{g}_\pi(z)$. By Lemma 12, for each $\tilde{z} \in \mathbb{T}^{4N+1-|\pi|}$, there exists a multi-index $\alpha = \alpha(\tilde{z})$ with $2 \leq |\alpha| \leq 3$ such that

$$|\partial^\alpha F_\pi(\tilde{z}, w, v)| = v|\partial^\alpha \tilde{g}_\pi(\tilde{z})| > \delta v,$$

for some $\delta > 0$. By Remark 7, we can take $u := vt \geq 1$ and $s := wt \geq 1$. Using a partition of unity, $|\alpha| := k = 3$ is the worst case for $vt \geq 1$. Applying Lemma 19 with rescaling and translating, (4.10) and (4.11) follow. □

The next two lemmas show that (4.10) and (4.11) hold when v is replaced by w.

Definition 17 Let $\mathfrak{P}_1$ denote the collection of partitions $\pi = \{S_1(\pi), \ldots, S_m(\pi)\}$ with properties:

(i) N and $N + 1$ do not belong to the same block;
(ii) $3N$ and $3N + 1$ do not belong to the same block;
(iii) $|S_k(\pi)| > 1$ for each $k = 1, \ldots, m$;
(iv) $N + 1, \ldots, 2N$ and $3N + 1, \ldots, 4N$ are non-minimal elements.

Remark 19 $\mathfrak{P}$ and $\mathfrak{P}_1$ are not the same subcollection of partitions. Their relation will be clear in the next lemma.

Lemma 13 *Suppose* $\pi \in \mathfrak{P}_1$. *There exist multi-indices* α *and* β *with* $|\alpha| = 2$, *and* $|\beta| = 3$ *such that* (4.8) *and* (4.9) *hold.*

Proof Let $\pi \in \mathfrak{P}_1$, and define

$$\mathcal{M}_\pi := \{j : 2N+1 \le j \le 4N-1 \text{ such that } j \text{ is non-maximal}\}.$$

Claim: $\mathcal{M}_\pi$ is nonempty. Suppose $\mathcal{M}_\pi$ is empty, then each j with $2N+1 \le j \le 4N$ is a maximal element, which implies that $|\pi| \ge 2N$. If $|\pi| > 2N$, then π has a block of single element, which means that π cannot belong to $\mathfrak{P}_1$. If $|\pi| = 2N$, then either all blocks of π have two elements or π has a block of a single element. We only need to consider when all blocks of π have two elements, in which case $2N+1, \dots, 4N$ are all maximal elements, and $1, \dots, 2N$ are all minimal elements, which contradicts the property (iv) in Definition 17. Therefore, $\mathcal{M}_\pi$ is nonempty for each $\pi \in \mathfrak{P}_1$ as claimed.

Let $p := \max\{j : j \in \mathcal{M}_\pi\}$. Then, ζ_p is one of the non-maximal variables on which $f_\pi(\zeta)$ does not depend. Hence,

$$\frac{\partial}{\partial z_p}\tilde{f}_\pi(z) = \frac{\partial}{\partial \zeta_p} f_\pi(\zeta) = 0,$$

for each $z \in \mathbb{T}^{4N+1-|\pi|}$. It remains to show that there exist multi-indices α, β, with $|\alpha| = 2$, $|\beta| = 3$ and $\alpha_p, \beta_p \ge 1$ such that (4.8) holds.

To that end, we note that $p+1$ is a maximal element. Since $|S_k(\pi)| > 1$ for each $k = 1, \dots, |\pi|$, there exists a non-maximal element $l \in \{1, \dots, p\}$ such that the non-maximal variable ζ_l does not appear in κ_i for each $i \ge p+1$, see also Remark 18(ii). However, κ_p depends on ζ_l and ζ_p.

If $p = 3N$, then ζ_{3N} is one of the non-maximal variables on which κ_{3N+1} depends because $3N$ and $3N+1$ do not belong to the same block, see also Remark 18(iii). Since $3N+2$ is a maximal element, there exists a non-maximal element $l \in \{1, \dots, 3N\}$ such that the non-maximal variable ζ_l does not appear in κ_i for each $i \ge 3N+2$. Then,

$$\frac{\partial^2}{\partial z_l \partial z_{3N}}\tilde{g}_\pi(z) = \frac{\partial^2}{\partial \zeta_l \partial \zeta_{3N}} g_\pi(\zeta) = \cos(\kappa_{3N+1})\Big|_z$$

and

$$\frac{\partial^3}{\partial z_l \partial z_{3N}^2}\tilde{g}_\pi(z) = \frac{\partial^3}{\partial \zeta_l \partial \zeta_{3N}^2} g_\pi(\zeta) = \sin(\kappa_{3N+1})\Big|_z$$

satisfy (4.8).

Now suppose $p \ne 3N$. We claim that $\cos(\kappa_p)$ is present in (4.4). For $p > 3N$, the claim follows directly since $\cos(\kappa_p)$ appears in the second sum in (4.4) and κ_p is distinct from any κ_i for $i < p$. For $p < 3N$, we will prove that $\cos(\kappa_p) \not\equiv \cos(\kappa_q)$ for any $q \ge 3N+1$. We note that q is a maximal element, so ζ_q is a maximal variable. Thus, there exists a non-maximal element $\nu \in \{1, \dots, p\}$ such that the non-maximal variable ζ_ν does not appear in κ_i for $i \ge q$. By (4.5), this means that κ_p depends on ζ_ν, but κ_q does not. Hence, $\kappa_p \ne \kappa_q$ as variables in $\mathbb{T}$. Therefore, $\cos(\kappa_p) \not\equiv \cos(\kappa_q)$, which proves the claim that $\cos(\kappa_p)$ is present in (4.4). Consequently,

$$\frac{\partial^2}{\partial z_l \partial z_p}\tilde{g}_\pi(z) = \frac{\partial^2}{\partial \zeta_l \partial \zeta_p} g_\pi(\zeta) = \mp \cos(\kappa_p)\Big|_z$$

and

$$\frac{\partial^3}{\partial z_l \partial z_p^2}\tilde{g}_\pi(z) = \frac{\partial^3}{\partial \zeta_l \partial \zeta_p^2} g_\pi(\zeta) = \mp \sin(\kappa_p)\Big|_z$$

satisfy (4.8). (The sign is a minus if $p < 3N$, and a plus if $p > 3N$.) This completes the proof of the lemma. □

Lemma 14 *Let* $\pi \in \mathfrak{P}$. *Then, with* $k = 3$ *and* $wt \geq 1$,

$$|\Xi_\pi(wt, vt)| \leq D_1 t^{-1/k} w^{-1/k}, \tag{4.12}$$

$$|\Lambda_\pi(wt, vt)| \leq D_2 t^{-1/k} w^{-1/k}, \tag{4.13}$$

where D_1 *and* D_2 *are constants independent of* w, v, *and* t.

Proof Before integrating over the delta functions, $\Xi_\pi(s,u)$ and $\Lambda_\pi(s,u)$ (see (3.50) and (3.51)) are integrals of the same form given by

$$\int_{\mathbb{T}^{4N+1}} e^{isf(\zeta)} e^{iug(\zeta)} \psi(\zeta) \tilde{K}_\pi(\zeta_\sharp)\, d\zeta,$$

where the phases $f(\zeta)$ and $g(\zeta)$ are given by

$$f(\zeta) = \sum_{j=1}^{N-1} \cos(\zeta_0 + \ldots + \zeta_j) - \sum_{j=N+1}^{2N-1} \cos(\zeta_0 + \ldots + \zeta_j), \tag{4.14}$$

$$g(\zeta) = \sum_{j=2N+1}^{3N-1} \cos(\zeta_0 + \ldots + \zeta_j) - \sum_{j=3N+1}^{4N-1} \cos(\zeta_0 + \ldots + \zeta_j). \tag{4.15}$$

Here, $\zeta_j \in \mathbb{T}$ for each $j = 0, \ldots, 4N$, and $\zeta = (\zeta_0, \zeta_1, \ldots, \zeta_{4N}) = (\zeta_0, \zeta_\sharp)$. The only difference between $\Xi_\pi(s,u)$ and $\Lambda_\pi(s,u)$ lies in the integrand $\psi(\zeta)$; namely, $\psi(\zeta) \equiv 1$ for $\Lambda_\pi(s,u)$, whereas $\psi(\zeta) := \sin(\zeta_0 + \ldots + \zeta_N)\sin(\zeta_0 + \ldots + \zeta_N + \ldots + \zeta_{3N})$ for $\Xi_\pi(s,u)$. With $s := wt$ and $u := vt$, it suffices to show that, for each $\pi \in \mathfrak{P}$,

$$\left| \int_{\mathbb{T}^{4N+1}} e^{iwtf(\zeta)} e^{ivtg(\zeta)} \psi(\zeta) \tilde{K}_\pi(\zeta_\sharp)\, d\zeta \right| \leq c(wt)^{-1/k},$$

for $wt \geq 1$ and some constant $c > 0$, where $\psi(\zeta)$ is any one of the two functions mentioned above. Recall that, for each $\pi \in \mathcal{P}$, $\zeta_1 + \ldots + \zeta_{4N} = 0$. We introduce new variables $w_0 = -\zeta_0$ and $w_j := \zeta_{4N-j+1}$, or equivalently, $\zeta_j = w_{4N-j+1}$, for $j = 1, \ldots, 4N$. In a matrix notation, $w_\sharp^T = A\zeta_\sharp^T$, where $\zeta_\sharp^T$ denotes a column vector of $\zeta_1, \ldots, \zeta_{4N}$, and $w_\sharp^T$ a column vector of $w_1, \ldots, w_{4N}$. It is easy to see that A is a $4N \times 4N$ matrix with one on the anti-diagonal (i.e., the diagonal going from the lower left corner to the upper right corner) and zero elsewhere. Thus, $\det(A) = 1$ and $A^2 = I$.

Consider $f(\zeta)$ in (4.14) and $g(\zeta)$ in (4.15). Rewriting the sum in (4.14) in reverse order and using $\zeta_1 + \ldots + \zeta_{4N} = 0$, $f(\zeta)$ can be expressed as

$$
\begin{aligned}
f(\zeta) &= -\cos(\zeta_0 + \zeta_1 + \ldots + \zeta_{2N-1}) - \ldots - \cos(\zeta_0 + \zeta_1 + \ldots + \zeta_{N-1} + \zeta_N + \zeta_{N+1}) \\
&\quad + \cos(\zeta_0 + \zeta_1 + \ldots + \zeta_{N-1}) + \ldots + \cos(\zeta_0 + \zeta_1) \\
&= -\cos(\zeta_0 - \zeta_{2N} - \ldots - \zeta_{4N}) - \ldots - \cos(\zeta_0 - \zeta_{N+2} - \ldots - \zeta_{4N}) \\
&\quad + \cos(\zeta_0 - \zeta_N - \ldots - \zeta_{4N}) + \ldots + \cos(\zeta_0 - \zeta_2 - \ldots - \zeta_{4N}) \\
&= -\cos(-w_0 - w_{2N+1} - \ldots - w_1) - \ldots - \cos(-w_0 - w_{3N-1} - w_{3N-2} - \ldots - w_1) \\
&\quad + \cos(-w_0 - w_{3N+1} - \ldots - w_1) - \ldots - \cos(-w_0 - w_{4N-1} - w_{4N-2} - \ldots - w_1) \\
&= -\cos(w_0 + w_1 + \ldots + w_{2N+1}) - \ldots - \cos(w_0 + w_1 + \ldots + w_{3N-1}) \\
&\quad + \cos(w_0 + w_1 + \ldots + w_{3N+1}) + \ldots + \cos(w_0 + w_1 + \ldots + w_{4N-1}) \\
&:= f^{\natural}(w).
\end{aligned}
$$

Similarly, we can do the same for $g(\zeta)$ in (4.15).

$$
\begin{aligned}
g(\zeta) &= \cos(\zeta_0 + \zeta_1 + \ldots + \zeta_{2N+1}) + \ldots + \cos(\zeta_0 + \zeta_1 + \ldots + \zeta_{3N-1}) \\
&\quad - \cos(\zeta_0 + \zeta_1 + \ldots + \zeta_{3N+1}) - \ldots - \cos(\zeta_0 + \zeta_1 + \ldots + \zeta_{4N-1}) \\
&= -\cos(w_0 + w_1) - \ldots - \cos(w_0 + \ldots + w_{N-1}) \\
&\quad + \cos(w_0 + \ldots + w_{N+1}) + \ldots + \cos(w_0 + \ldots + w_{2N-1}) \\
&:= g^{\natural}(w).
\end{aligned}
$$

Indeed, $f^{\natural}(w) = -g(w)$ and $g^{\natural}(w) = -f(w)$; in particular, $f^{\natural}(w)$ depends on variables $w_0, w_1, \ldots, w_{4N}$, whereas $g^{\natural}(w)$ only depends on $w_0, w_1, \ldots, w_{2N-1}$. Moreover,

$$
\begin{aligned}
\sin(\kappa_N)\sin(\kappa_{3N}) &:= \sin(\zeta_0 + \ldots + \zeta_N)\sin(\zeta_0 + \ldots + \zeta_N + \ldots + \zeta_{3N}) \\
&= \sin(\zeta_0 - \zeta_{N+1} - \ldots - \zeta_{4N})\sin(\zeta_0 - \zeta_{3N+1} - \ldots - \zeta_{4N}) \\
&= \sin(-w_0 - w_{3N} - \ldots - w_1)\sin(-w_0 - w_N - \ldots - w_1) \\
&= \sin(w_0 + w_1 + \ldots + w_N)\sin(w_0 + w_1 + \ldots + w_{3N}).
\end{aligned}
$$

Thus, $\psi(\zeta) = \psi(w)$. Next, we consider the effect of the change of variables $\zeta_\sharp^T \mapsto w_\sharp^T := A\zeta_\sharp^T$ on $\tilde{K}_\pi(\zeta_\sharp)$. Since the constraint equations, obtained by setting each argument in the delta functions equal zero, are given by a homogeneous system of linear equations, where each ζ_j, for $1 \le j \le 4N$, appears exactly once with coefficient one, we can write $B\zeta_\sharp^T = 0$ for some $|\pi| \times 4N$ matrix B, where each entry of B is either zero or one. In terms of the new variable $w_\sharp^T$, the constraint equations are given by $BAw_\sharp^T := \tilde{B}w_\sharp^T = 0$. In effect, A reverses the order of the columns of B. This means that if ζ_j is a maximal variable, then the corresponding w_{4N-j+1} is a minimal variable, and if ζ_j is a non-maximal variable, then w_{4N-j+1} is a non-minimal variable associated with π. Then, $\tilde{K}_\pi(\zeta_\sharp) = \tilde{K}_\pi((Aw_\sharp^T)^T) := \tilde{\tilde{K}}_\pi(w_\sharp)$, where the new kernel $\tilde{\tilde{K}}_\pi(w_\sharp)$ consists of a product of delta functions whose arguments are linear combinations of w_j, where each w_j, for $1 \le j \le 4N$, appears exactly once with coefficient one.

Claim: If $\pi \in \mathfrak{P}$, then $\tilde{\tilde{K}}_\pi(w_\sharp) = \tilde{K}_{\tilde{\pi}}(w_\sharp)$ for some $\tilde{\pi} \in \mathfrak{P}_1$. (See Definitions 16 and 17.)

Proof of Claim: If $\pi \in \mathfrak{P}$, then $1, \ldots, N$ and $2N+1, \ldots, 3N$ are non-maximal elements, which means that $\zeta_1, \ldots, \zeta_N$ and $\zeta_{2N+1}, \ldots, \zeta_{3N}$ are non-maximal variables, hence $w_{N+1}, \ldots, w_{2N}$ and $w_{3N+1}, \ldots, w_{4N}$ are non-minimal variables. Moreover, w_{4N-j+1} and w_{4N-l+1} appear in the same constraint equation corresponding to $\tilde{\tilde{K}}_\pi(w_\sharp)$ if and only if ζ_j and ζ_l appear in the same constraint equation corresponding to $\tilde{K}_\pi(\zeta_\sharp)$. This implies that the above change of variables preserves the properties (i)–(iii) in Definition 16. This completes the proof of the claim.

Consequently, if $\pi \in \mathfrak{P}$,

$$\begin{aligned}\int_{\mathbb{T}^{4N+1}} e^{isf(\zeta)} e^{iug(\zeta)} \psi(\zeta) \tilde{K}_\pi(\zeta_\sharp)\, d\zeta &= \int_{\mathbb{T}^{4N+1}} e^{isf^\natural(w)} e^{iug^\natural(w)} \psi(w) \tilde{\tilde{K}}_\pi(w_\sharp)\, dw \\ &= \int_{\mathbb{T}^{4N+1}} e^{-isg(w)} e^{-iuf(w)} \psi(w) \tilde{K}_{\tilde{\pi}}(w_\sharp)\, dw,\end{aligned}$$

for some $\tilde{\pi} \in \mathfrak{P}_1$. To obtain the upper bounds in (4.12) and (4.13), we apply Lemma 13 and follow the proof of Corollary 3, where w and v are interchanged. □

To summarize, we have shown that if $\pi \in \mathfrak{P}$, $wt \geq 1$ and $vt \geq 1$, then $\Xi_\pi(wt, vt)$ satisfies the inequalities (4.10) and (4.12), and $\Lambda_\pi(wt, vt)$ satisfies (4.11) and (4.13). Thus, $\mathbb{I}(wt, vt) := \Xi_\pi(wt, vt)\Lambda_\pi(wt, vt)^{d-1}$ satisfies (B.10). Then, we follow Corollary 7 to obtain the following result.

Corollary 4 *Let $\pi \in \mathfrak{P}$. Then, for large t,*

$$\int_1^t \int_1^t |\Xi_\pi(s,u)||\Lambda_\pi(s,u)|^{d-1}\, ds du = \begin{cases} O(t^{2-\frac{d}{k}}) & \text{if } d < 2k, \\ O((\ln t)^2) & \text{if } d = 2k, \\ O(1) & \text{if } d \geq 2k. \end{cases} \tag{4.16}$$

In particular, combined with Remark 7, this implies that there exists a constant $C_\pi > 0$ such that

$$|\Gamma_\pi(t)| \leq \int_0^t \int_0^t |\Xi_\pi(s,u)||\Lambda_\pi(s,u)|^{d-1}\, ds du \leq C_\pi\, t,$$

when $d \geq k = 3$.

Proof Let $\pi \in \mathfrak{P}$ be fixed. With $k = 3$, $s = wt \geq 1$ and $u = vt \geq 1$,

$$|\Xi_\pi(wt, vt)||\Lambda_\pi(wt, vt)|^{d-1} \leq C t^{-d/k} w^{-d\vartheta/k} v^{-d(1-\vartheta)/k},$$

for any $\vartheta \in [0, 1]$ and some constant $C > 0$ independent of w, v and t. Then,

$$\begin{aligned}\int_1^t \int_1^t |\Xi_\pi(s,u)||\Lambda_\pi(s,u)|^{d-1}\, ds du &\leq t^2 \int_{1/t}^1 \int_{1/t}^1 |\Xi_\pi(wt, vt)||\Lambda_\pi(wt, vt)|^{d-1}\, dw dv \\ &\leq C t^2 t^{-d/k} \int_0^1 \int_0^1 w^{-d\vartheta/k} v^{-d(1-\vartheta)/k}\, dw dv.\end{aligned}$$

Therefore, the claimed statements follows from the estimates in Corollary 7. □

5 Proof of the Main Result

Finally, we put all pieces together to prove Proposition 1. The key strategy is very simple. For each $\pi \in \mathcal{P}$, we will show that $|A_\pi(t)| = O_N(t)$ for large t in dimensions $d \geq 3$ provided $\check{\mu} \in L^2(\mathbb{R})$. The following lemmas give estimates of $A_\pi(t)$ for each $\pi \in \mathfrak{P}$.

Lemma 15 *Suppose $d \geq 3$. If $\pi \in \mathfrak{P}$, then there exists a constant $C_\pi(N)$ such that $|A_\pi(t)| \leq C_\pi(N)t$.*

Proof For $\pi \in \mathfrak{P}$, by (3.54),

$$\begin{aligned}|A_\pi(t)| &:= \left| \int_0^t \int_0^t \hat{\mathcal{U}}_\pi(s,u) \Xi_\pi(s,u) \Lambda_\pi(s,u)^{d-1}\, ds du \right| \\ &\leq (4d) \sup_{(s,u)\in[0,t]\times[0,t]} |\check{\mathcal{U}}_\pi(s,u)| \int_0^t \int_0^t |\Xi_\pi(s,u)||\Lambda_\pi(s,u)|^{d-1}\, ds du \\ &\leq C_\pi(N)t,\end{aligned}$$

by Corollary 4 with $k := 3$, which completes the proof for $d \geq 3$. □

Lemma 16 *Suppose $\check{\mu} \in L^2(\mathbb{R})$. If $\pi \notin \mathcal{P}_0$, then there exists a constant $C_\pi(N)$ such that*

$$\int_0^t \int_0^t |\check{\mathcal{U}}_\pi(s,u)|\, ds du \leq C_\pi(N) \|\check{\mu}\|_2^2\, t. \tag{5.1}$$

Proof By (3.35),

$$|\check{\mathcal{U}}_\pi(s,u)| \leq \sum_{\pi' \prec \pi} |n_{\pi,\pi'}||\check{\mu}_{\pi'}(s,u)|.$$

If $\pi \notin \mathcal{P}_0$, then, by Lemma 11 and Corollary 2, each term $|\check{\mu}_{\pi'}(s,u)|$ in the above sum depends on s or u. Specifically, for each $\pi' \prec \pi \notin \mathcal{P}_0$,

$$|\check{\mu}_{\pi'}(s,u)| \leq |\check{\mu}(v)||\check{\mu}(v')|,$$

for some $v = a_1 s/N + a_2 u/N$ and $v' = a_3 s/N + a_4 u/N$, where a_1, a_2, a_3, and a_4 are integers with $|a_1| + |a_2| \geq 1$ and $|a_3| + |a_4| \geq 1$. Then,

$$\int_0^t \int_0^t |\check{\mu}(v)||\check{\mu}(v')|\, ds du = N^2 \int_0^{t/N} \int_0^{t/N} |\check{\mu}(a_1 s + a_2 u)||\check{\mu}(a_3 s + a_4 u)|\, ds du. \tag{5.2}$$

Without loss of generality, we can assume $a_1 \neq 0$. Let $w = u$, $y = a_1 s + a_2 u$. Then,

$$\begin{aligned}&\int_0^{t/N} \int_0^{t/N} |\check{\mu}(a_1 s + a_2 u)||\check{\mu}(a_3 s + a_4 u)|\, ds du \\ &\quad \leq C_1 \int_0^{t/N} \int_{-\beta t/N}^{\beta t/N} |\check{\mu}(by + cw)||\check{\mu}(y)|\, dy dw,\end{aligned}$$

for some $C_1, \beta > 0$ and some constants b, c. If $c = 0$, then we apply the Schwarz inequality. If $c \neq 0$, we first integrate over w and apply the Schwarz inequality. In both cases, we can conclude that

$$\int_0^t \int_0^t |\check{\mu}(v)||\check{\mu}(v')|\, ds du \le C_2(N)\|\check{\mu}\|_2^2\, t,$$

for some $C_2(N) > 0$, which proves the lemma. □

Corollary 5 *Suppose* $\check{\mu} \in L^2(\mathbb{R})$. *If* $\pi \in \mathcal{P} \setminus \mathfrak{P}$, *then either* $\Xi_\pi(s,u) \equiv 0$ *or there exists a constant* $C_\pi(N)$ *such that*

$$\int_0^t \int_0^t |\check{\mathcal{U}}_\pi(s,u)|\, ds du \le C_\pi(N)\|\check{\mu}\|_2^2\, t.$$

Proof If $\pi \in \mathcal{P} \setminus \mathfrak{P}$, then π fails to satisfy at least one of the properties (i)–(iv) in Definition 16. If Definition 16(i) or Definition 16(ii) is not satisfied, then $\Xi_\pi(s,u) \equiv 0$ by Lemma 9.

If Definition 16(iii) is not satisfied, then π contains a block of single element, so π is not even, and thus, $\pi \notin \mathcal{P}_0$. Similarly, if Definition 16(iv) is not satisfied, then there exists at least one maximal element $j \in \{1, \dots, N, 2N+1, \dots, 3N\}$. This implies that there exists a block of π, say $S \subset \mathfrak{S}$, such that $\sum_{j\in S} t_j \neq 0$; thus, $\pi \notin \mathcal{P}_0$. Consequently, the estimate in Lemma 16 is valid in the latter two cases. □

Proof of Proposition 1 Recall the definition of the amplitude $A_\pi(t)$ in (3.54):

$$A_\pi(t) = (4d) \int_0^t \int_0^t \hat{\mathcal{U}}_\pi(s,u)\Xi_\pi(s,u)\Lambda_\pi(s,u)^{d-1}\, ds du.$$

Since $\mathbb{E}_\omega\|\mathbf{X}_N(t)\delta_{\mathbf{0}}\|^2 \le \sum_{\pi\in\mathcal{P}} |A_\pi(t)| = \sum_{\pi\in\mathfrak{P}} |A_\pi(t)| + \sum_{\pi\in\mathcal{P}\setminus\mathfrak{P}} |A_\pi(t)|$, it suffices to show that $|A_\pi(t)| = O_N(t)$ for each $\pi \in \mathcal{P}$, in dimensions $d \ge 3$ when $\check{\mu} \in L^2(\mathbb{R})$. This is accomplished in Lemma 15 and Corollary 5, thus completing the proof of the main result. □

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Appendix A: Approximate Identity

Lemma 17 *For* $\epsilon > 0$, *let* $\phi_\epsilon(\kappa) := \sum_{\mathbf{n}\in\mathbb{Z}^d} e^{-i\kappa\cdot\mathbf{n}} e^{-\epsilon\|\mathbf{n}\|}$. *Then,* $\{\phi_\epsilon\}$ *is an approximated identity such that*

$$\lim_{\epsilon\downarrow 0} \int_{\mathbb{T}^d} \psi(\kappa)\phi_\epsilon(\kappa)\, d\kappa = \int_{\mathbb{T}^d} \psi(\kappa)\delta(\kappa)\, d\kappa, \tag{A.1}$$

for any $\psi \in C^\infty(\mathbb{T}^d)$. *Thus,*

$$\lim_{\epsilon\downarrow 0} \sum_{\mathbf{n}\in\mathbb{Z}^d} e^{-i\kappa\cdot\mathbf{n}} e^{-\epsilon\|\mathbf{n}\|} = \delta(\kappa), \tag{A.2}$$

in the sense of distributions. Here, $\kappa \in \mathbb{T}^d$, *and the Dirac delta function is thought of as having a period* 2π.

Proof Let $\psi \in C^\infty(\mathbb{T}^d)$. Then,

$$\begin{aligned}\lim_{\epsilon\downarrow 0}\int_{\mathbb{T}^d}\psi(\kappa)\sum_{\mathbf{n}\in\mathbb{Z}^d}e^{-i\kappa\cdot\mathbf{n}}e^{-\epsilon\|\mathbf{n}\|}\,d\kappa &= \lim_{\epsilon\downarrow 0}\sum_{\mathbf{n}\in\mathbb{Z}^d}e^{-\epsilon\|\mathbf{n}\|}\check{\psi}(\mathbf{n})\\ &= \sum_{\mathbf{n}\in\mathbb{Z}^d}\check{\psi}(\mathbf{n})\\ &= \psi(0)\\ &= \int_{\mathbb{T}^d}\psi(\kappa)\delta(\kappa)d\kappa. \end{aligned}$$

□

Appendix B: Oscillatory Integrals

In this section, we consider an oscillatory integral with two generic phases. Let

$$\mathbb{I}(s,u) := \int_{\mathbb{R}^n} e^{is\phi(\mathbf{x})}e^{iu\varphi(\mathbf{x})}\psi(\mathbf{x})\,d\mathbf{x}, \tag{B.1}$$

$$\Pi(t) := \int_0^t\int_0^t \mathbb{I}(s,u)\,dsdu, \tag{B.2}$$

where $\phi, \varphi \in C^\infty$ are arbitrary and $\psi \in C_c^\infty$ is a cut-off function.

Remark 20 If $s = 0, u = 0$ or $s = u$ in (B.1), then the integral is reduced to an ordinary oscillatory integral with one phase, which is very important and much studied due to its close relation to many problems in mathematics and sciences, see [37, 39] and references therein. In such a situation, many results are known, see, for example, [10, 20, 31, 37, 40]. Estimating an oscillatory integral with two phases, like that in (B.1), is more technical than that with one phase because the critical points of $s\phi(\mathbf{x}) + u\varphi(\mathbf{x})$ generally depend on s and u. To the author's knowledge, not much work, if any, has been done in this direction before.

Remark 21 For current purposes, it suffices to consider (B.2) and determine sufficient conditions on ϕ and φ such that $|\Pi(t)| \le ct$. In [39], the author obtains some general results by applying the Weierstrass Preparation theorem when ϕ and φ are analytic and at least one of $\nabla\phi, \nabla\varphi$ does not vanish, and by applying the Splitting lemma [5] when ϕ and φ are C^∞, and their Hessian matrices satisfy certain rank conditions. However, for $A_\pi(t)$ and $\Gamma_\pi(t)$, the phases $f_1(N, \boldsymbol{\zeta}_1)$ and $g_1(N, \boldsymbol{\zeta}_1)$ obtained after $\tilde{K}_\pi(\boldsymbol{\zeta}_{\sharp,1})$ is integrated out do not always satisfy such conditions because they can vanish identically, see Lemma 10. Even when $f_1(N, \boldsymbol{\zeta}_1)$ and $g_1(N, \boldsymbol{\zeta}_1)$ do not vanish identically, they and all of their first and second derivatives can vanish at the same point, see Example 4.3.

Lemma 18 [37] *Let ψ be a smooth function supported in the unit ball of $\mathbb{R}^n$, and ϕ be a real-valued function on $\mathbb{R}^n$. Suppose that, for some multi-index α with $|\alpha| = k \ge 1$, $|\partial_{\mathbf{x}}^\alpha\phi| \ge 1$ throughout the support of ψ. Then,*

$$\left|\int_{\mathbb{R}^n} e^{is\phi(\mathbf{x})}\psi(\mathbf{x})\,d\mathbf{x}\right| \le C_k(\phi)\big(\|\psi\|_\infty + \|\nabla\psi\|_1\big)s^{-1/k}, \tag{B.3}$$

where the constant $C_k(\phi)$ is independent of ψ and s, and remains bounded as long as the C^{k+1}-norm of ϕ remains bounded.

Proof The proof is given in [37] (Proposition VIII.5). □

B.1 Oscillatory Integral with Two Phases

Consider a special case of $\Pi(t)$ in (B.2) with $\mathbb{I}(s,u) = \prod_{l=1}^{d} I_l(s,u)$, where

$$I_l(s,u) := \int_{\mathbb{R}^n} e^{is\phi(\mathbf{x})} e^{iu\varphi(\mathbf{x})} \psi_l(\mathbf{x})\, d\mathbf{x}, \tag{B.4}$$

in which ψ_l is a cut-off function for each $l = 1, \dots, d$ and at our disposal. Put $s = tw$, $u = tv$. Then,

$$\Pi(t) := \int_0^t \int_0^t \mathbb{I}(s,u)\, ds du = t^2 \int_0^1 \int_0^1 \prod_{l=1}^{d} I_l(wt, vt)\, dw dv. \tag{B.5}$$

It suffices to first estimate $I_l(wt, vt)$ for each $l = 1, \dots, d$, and then integrate with respect to w and v to obtain an upper bound for $\Pi(t)$.

Let $F : \mathbb{R}^n \times [0,1] \times [0,1] \to \mathbb{R}$ be defined as $F(\mathbf{x}, w, v) := w\phi(\mathbf{x}) + v\varphi(\mathbf{x})$, where ϕ and φ are smooth functions. (Note that the phases in our problem are analytic.) Then,

$$I_l(wt, vt) = \int_{\mathbb{R}^n} e^{itF(\mathbf{x},w,v)} \psi_l(\mathbf{x})\, d^n\mathbf{x}. \tag{B.6}$$

Define

$$\|F\|^{\bullet}_{C^{k+1}} := \sup_{(w,v)\in[0,1]\times[0,1]} \|F(\cdot, w, v)\|_{C^{k+1}},$$

where $\|F(\cdot, w, v)\|_{C^{k+1}}$ denotes the C^{k+1}-norm of $F(\mathbf{x}, w, v)$ with respect to $\mathbf{x}$ for fixed $(w, v) \in [0,1] \times [0,1]$. It is clear that $\|F\|^{\bullet}_{C^{k+1}} \le \|\phi\|_{C^{k+1}} + \|\varphi\|_{C^{k+1}}$.

Lemma 19 *Consider* (B.6). *Let ψ_l be a smooth function supported in the unit ball of $\mathbb{R}^n$. If there exists a multi-index α with $|\alpha| \ge 1$ such that $|\partial_{\mathbf{x}}^{\alpha} F(\mathbf{x}, w, v)| \ge \delta w > 0$ on the support of ψ_l, then*

$$|I_l(wt, vt)| \le C_\alpha \big(\|\psi_l\|_\infty + \|\nabla\psi_l\|_1\big)(\delta w t)^{-1/|\alpha|}. \tag{B.7}$$

Similarly, if there exists a multi-index β with $|\beta| \ge 1$ such that $|\partial_{\mathbf{x}}^{\beta} F(\mathbf{x}, w, v)| \ge \delta v > 0$ on the support of ψ_l, then

$$|I_l(wt, vt)| \le D_\beta \big(\|\psi_l\|_\infty + \|\nabla\psi_l\|_1\big)(\delta v t)^{-1/|\beta|}. \tag{B.8}$$

Here, the constants C_α and D_β are independent of w, v, and t, and remain finite as long as $\|F\|^{\bullet}_{C^{|\alpha|+1}}$ and $\|F\|^{\bullet}_{C^{|\beta|+1}}$ remain finite on the support of ψ_l.

Proof Apply Lemma 18. □

Remark 22 If ψ_l is supported in any compact set of $\mathbb{R}^n$ and $|\partial_{\mathbf{x}}^{\alpha} F(\mathbf{x}, w, v)| \ge \delta w > 0$ (resp. $|\partial_{\mathbf{x}}^{\beta} F(\mathbf{x}, w, v)| \ge \delta v > 0$) on its support, then Lemma 19 also applies by rescaling and translating ψ_l, which do not affect the decay estimates $(\delta w t)^{-1/|\alpha|}$ (resp. $(\delta v t)^{-1/|\beta|}$).

Corollary 6 *If* (B.7) *and* (B.8) *hold, then*

$$\left|I_l(wt, vt)\right| \leq \mathcal{E}_l t^{-1/|\beta|+\vartheta(1/|\beta|-1/|\alpha|)} w^{-\vartheta/|\alpha|} v^{-(1-\vartheta)/|\beta|}, \tag{B.9}$$

for any $\vartheta \in [0, 1]$, *where* $\mathcal{E}_l$ *is independent of* w, v, *and* t, *and remains finite as long as* $\|F\|^{\bullet}_{C^{|\alpha|+1}}$ *and* $\|F\|^{\bullet}_{C^{|\beta|+1}}$ *remain finite.*

Consequently, an upper bound on $|\mathbb{I}(tw, tv)| = \prod_{l=1}^{d} |I_l(s, u)|$ is given by the product of the bound in (B.9). In our problem, we note that all factors $I_l(wt, vt)$ in $\mathbb{I}(tw, tv) := \prod_{l=1}^{d} I_l(wt, vt)$ have identical phases but may have different functions in the integrand, playing a similar role as that of ψ_l, see Lemma 8. This is in accordance with the definitions of $\Xi_\pi(s, u)$ and $\Lambda_\pi(s, u)$ in (3.50) and (3.51). When $|\alpha| = |\beta| = k$, then

$$\left|\mathbb{I}(tw, tv)\right| \leq C t^{-d/k} w^{-d\vartheta/k} v^{-d(1-\vartheta)/k}, \tag{B.10}$$

for some constant $C > 0$ independent of t, w, and v.

Corollary 7 *Suppose* $\mathbb{I}(wt, vt)$ *satisfies* (B.10). *Then, for large* t,

$$|\Pi(t)| = \begin{cases} O(t^{2-\frac{d}{k}}) & \text{if } d < 2k, \\ O((\ln t)^2) & \text{if } d = 2k, \\ O(1) & \text{if } d \geq 2k. \end{cases} \tag{B.11}$$

Proof If $d/2k < 1$, we choose $\vartheta = 1/2$ in (B.10) so that

$$\gamma := \int_0^1 \int_0^1 w^{-d/2k} v^{-d/2k}\, dw dv < \infty.$$

Thus, by (B.5) and (B.10),

$$|\Pi(t)| \leq \gamma C t^{2-d/k},$$

which proves the claimed statement for $d < 2k$.

For $d \geq 2k$, we also notice that $|\mathbb{I}(tw, tv)| \leq c$ for some constant $c > 0$ independent of w, v and t. Then, for any $0 < \eta < 1$, we can write

$$\begin{aligned}\int_0^1 \int_0^1 \mathbb{I}(tw, tv)\, dw dv &= \int_0^\eta \int_0^\eta \mathbb{I}(tw, tv)\, dw dv + \int_0^\eta \int_\eta^1 \mathbb{I}(tw, tv)\, dv dw \\ &\quad + \int_0^\eta \int_\eta^1 \mathbb{I}(tw, tv)\, dw dv + \int_\eta^1 \int_\eta^1 \mathbb{I}(tw, tv)\, dw dv.\end{aligned}$$

The first integral is bounded by $c\eta^2$. By interchanging the roles of ϑ and $1 - \vartheta$ in (B.10), the second and third integrals can be estimated by the same bound. By (B.10) and choosing $\vartheta = 1$,

$$\begin{aligned}\left|\int_0^\eta \int_\eta^1 \mathbb{I}(tw, tv)\, dw dv\right| &\leq C t^{-d/k} \int_0^\eta \int_\eta^1 w^{-d\vartheta/k} v^{-d(1-\vartheta)/k}\, dw dv \\ &\leq C_1 t^{-d/k} \eta^{2-d/k}.\end{aligned}$$

Moreover, with $\vartheta = 1/2$,

$$\left|\int_\eta^1 \int_\eta^1 \mathbb{I}(tw, tv)\, du dv\right| \le Ct^{-d/k} \int_\eta^1 \int_\eta^1 w^{-d/2k} v^{-d/2k}\, dw dv$$

$$\le C_2 t^{-d/k} \begin{cases} (\ln \eta)^2 & \text{if } d = 2k, \\ \eta^{2-d/k} & \text{if } d > 2k. \end{cases}$$

All estimates hold for any $0 < \eta < 1$. Thus, for t large, optimizing over η yields

$$\left|\int_0^1 \int_0^1 \mathbb{I}(tw, tv)\, dw dv\right| \le C_3 \begin{cases} t^{-2}(\ln(t))^2 & \text{if } d = 2k, \\ t^{-2} & \text{if } d > 2k, \end{cases}$$

for some $C_3 > 0$. Together with (B.5), this completes the proof of the corollary. □

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